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M338 Topology

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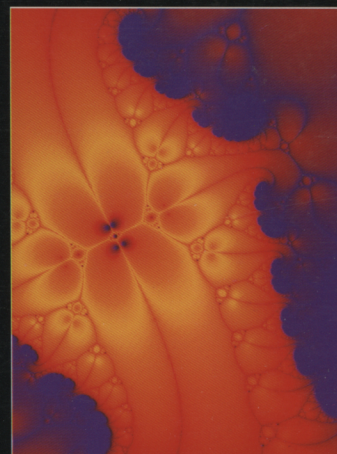
Unit A4 Closed sets

The Open University

A4

Closed sets

Contents

Introduction

At this stage of your study you have made considerable progress towards the goal of understanding continuity: you now have a definition of continuity for the general setting of topological spaces, and you know that this includes the class of metric spaces — those topological spaces for which there is a notion of distance.

You have also seen the crucial role played by open sets in the definitions of topological spaces and of continuity for such spaces. In this unit we investigate a class of sets that are closely related to open sets: the complements of open sets, called *closed sets*. De Morgan's Laws enable us to find many of the properties of closed sets by looking at the corresponding properties of open sets. We shall also see that we can define continuity in terms of closed sets.

De Morgan's Laws are given in *Unit A3*, Theorem 2.10.

This unit also discusses the structure of sets in topological spaces. We shall see how the concept of a *neighbourhood* of a point allows us to create a closed set from any set. The concept of a neighbourhood will also allow us to define the *boundary*, *interior* and *exterior* of a set.

This unit completes our initial discussion of topological spaces and gives a framework within which to place our examination of surfaces in Block B.

We return to a more general discussion of topological spaces in Block C.

Study guide

In Section 1, *Introducing closed sets*, we define a *closed set*, and show how many results for open sets have counterparts for closed sets. You should study this section carefully.

In Section 2, *Closure*, we introduce the notion of *closure* for a set. In order to do this, we first define the *neighbourhood* of a point.

Section 3, *Dense sets and isolated points*, is a short section that looks at how some sets can be *dense* in a topological space and at how some sets can have *isolated points*.

Section 4, *Interiors, exteriors and boundaries*, contains the definitions of the *interior*, *exterior* and *boundary* of a set. These are important concepts and you should take time to ensure that you can find the interior, exterior and boundary of simple sets.

There is no software associated with this unit.

1 Introducing closed sets

After working through this section, you should be able to:

▶ state and use the definition of a *closed set*;

▶ show that a closed ball, the complement of an open ball and a sphere in a metric space are closed sets;

▶ discuss the nature of closed sets for some simple topologies, including the subspace topology;

▶ state the definition of a topology in terms of closed sets;

▶ characterize a continuous function in terms of closed sets.

You are already familiar with some sets that are described as closed. For example, the interval $[a, b]$ is a *closed* interval. In this section, we define a closed subset of a general topological space. In order to do this, we first look at the properties of those sets that we already think of as being closed and describe these in terms of open sets.

Having defined closed sets, we then look at examples of closed sets for many of the topological spaces that you met in the previous unit, and also at some of the properties of closed sets.

1.1 What is a closed set?

We begin our study of closed sets by considering subsets of the real line. Consider the intervals (a, b), $[a, b]$, $[a, b)$ and $(a, b]$. How do we distinguish between them? Clearly, it is the inclusion or exclusion of the endpoints that distinguishes the intervals. The only *closed* interval is $[a, b]$ — the only interval that contains both its endpoints.

Now suppose that we are allowed to use only the open subsets of $\mathbb{R}$ (with the Euclidean topology) to identify which of these intervals is closed. How can we do it? We might begin by observing that (a, b) is the only open set; this, however, does not distinguish between the intervals $[a, b]$, $[a, b)$ and $(a, b]$. Now suppose that we look at the *complements* of the four intervals.

The complement of (a, b) with respect to $\mathbb{R}$ is

$$(a, b)^c = \mathbb{R} - (a, b) = (-\infty, a] \cup [b, \infty),$$

which is not an open set.

The complement of $[a, b]$ with respect to $\mathbb{R}$ is

$$[a, b]^c = \mathbb{R} - [a, b] = (-\infty, a) \cup (b, \infty),$$

which is the union of two open intervals, and so is an open set.

The complement of $[a, b)$ with respect to $\mathbb{R}$ is

$$[a, b)^c = \mathbb{R} - [a, b) = (-\infty, a) \cup [b, \infty),$$

which is not an open set.

Finally, the complement of $(a, b]$ with respect to $\mathbb{R}$ is

$$(a, b]^c = \mathbb{R} - (a, b] = (-\infty, a] \cup (b, \infty),$$

which is not an open set.

Recall, from *Unit A3*, that, if A is a subset of X, the *complement* of A with respect to X is

$$A^c = X - A$$
$$= \{x \in X : x \notin A\}.$$

So, $[a, b]$ is the only one of these four intervals whose complement with respect to $\mathbb{R}$ is an open set. Thus we have a method, using only open sets, of distinguishing between the closed interval $[a, b]$ and the intervals (a, b), $[a, b)$ and $(a, b]$. This does not help us to pin down the properties of an endpoint, but it does enable us to distinguish between an interval that contains both its endpoints and one that does not.

Problem 1.1

(a) Write down the complement of A with respect to $\mathbb{R}$ when:

(i) $A = [0, \infty)$; (ii) $A = (0, 2]$; (iii) $A = [0, 1] \cup [2, 3]$.

(b) Which of these complements are open sets in $\mathbb{R}$ (with the Euclidean topology)?

Example 1.1

Let us now look at the plane with the Euclidean topology — that is, the topological space $(\mathbb{R}^2, \mathcal{T}(d^{(2)}))$. An analogue of an open interval (a, b) is an *open disc* $B(\mathbf{a}, r)$. This is open because all open discs are open sets for the Euclidean topology on $\mathbb{R}^2$. The analogue of a closed interval $[a, b]$ in $\mathbb{R}$ should perhaps be a *closed disc*

$$B[\mathbf{a}, r] = \{\mathbf{x} \in \mathbb{R}^2 : d^{(2)}(\mathbf{a}, \mathbf{x}) \leq r\}.$$

This is not an open set, because it does not have the fried-egg property: it comprises the open disc

$$B(\mathbf{a}, r) = \{\mathbf{x} \in \mathbb{R}^2 : d^{(2)}(\mathbf{a}, \mathbf{x}) < r\}$$

together with its 'boundary', the sphere

$$S(\mathbf{a}, r) = \{\mathbf{x} \in \mathbb{R}^2 : d^{(2)}(\mathbf{a}, \mathbf{x}) = r\};$$

and, for each point $\mathbf{b} \in S(\mathbf{a}, r)$, the open disc $B(\mathbf{b}, s)$ centred at $\mathbf{b}$ and of radius s contains some points not in $B[\mathbf{a}, r]$, no matter how small we take $s > 0$, as Figure 1.1 illustrates.

Having established that $B[\mathbf{a}, r]$ is not open, we consider its complement with respect to $\mathbb{R}^2$,

$$B[\mathbf{a}, r]^c = \mathbb{R}^2 - B[\mathbf{a}, r] = \{\mathbf{x} \in \mathbb{R}^2 : d^{(2)}(\mathbf{a}, \mathbf{x}) > r\}.$$

If $B[\mathbf{a}, r]$ is the planar equivalent of a closed interval, then we might expect its complement to be open. It is, as we now demonstrate.

Let $\mathbf{x} \in B[\mathbf{a}, r]^c$. We want to prove that a positive number t can be found so that, for all $\mathbf{y} \in B(\mathbf{x}, t)$, we have $d^{(2)}(\mathbf{a}, \mathbf{y}) > r$. If we succeed in showing this, it follows that $B(\mathbf{x}, t) \subseteq B[\mathbf{a}, r]^c$, which shows that $B[\mathbf{a}, r]^c$ has the fried-egg property, and so is an open set.

So, let $d^{(2)}(\mathbf{a}, \mathbf{x}) = s$ and let $\mathbf{y} \in B(\mathbf{x}, t)$. Note that $s > r$, since $\mathbf{x} \in B[\mathbf{a}, r]^c$.

By the Triangle Inequality and using $d^{(2)}(\mathbf{y}, \mathbf{x}) < t$, we have

$$d^{(2)}(\mathbf{a}, \mathbf{y}) + t > d^{(2)}(\mathbf{a}, \mathbf{y}) + d^{(2)}(\mathbf{y}, \mathbf{x}) \geq d^{(2)}(\mathbf{a}, \mathbf{x}) = s.$$

Therefore,

$$d^{(2)}(\mathbf{a}, \mathbf{y}) > s - t.$$

Now $s > r$, so we can choose t so that $s - t > r$. Thus, $d^{(2)}(\mathbf{a}, \mathbf{y}) > r$, and so $B(\mathbf{x}, t) \subseteq B[\mathbf{a}, r]^c$. Hence $B[\mathbf{a}, r]^c$ is open. ∎

The terms *open disc* and *closed disc* are commonly used as alternatives to open ball and closed ball when the topological space is $(\mathbb{R}^2, \mathcal{T}(d^{(2)}))$.

In this subsection, for simplicity, we omit the subscript $d^{(2)}$ from the notation for open and closed balls/discs, etc.
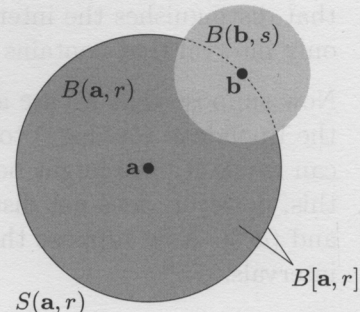
Figure 1.1
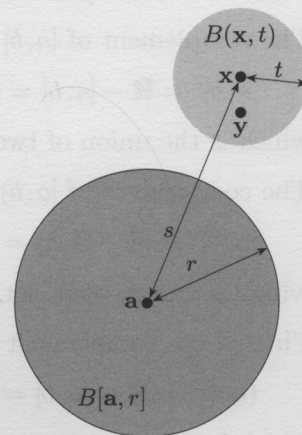
Figure 1.2

Problem 1.2

Show that the complement with respect to $\mathbb{R}^2$ of the closed square

$$S = [-1, 1] \times [-1, 1]$$

is open in $\mathbb{R}^2$ (with the Euclidean topology).

It appears that the sets that we think of as closed subsets of $\mathbb{R}$ and $\mathbb{R}^2$ (with the Euclidean topology) have complements that are open. This motivates our definition of a closed subset of a general topological space $(X, \mathcal{T})$.

Definition

Let $(X, \mathcal{T})$ be a topological space.

A set $A \subseteq X$ is **closed** if $A^c = X - A$ is open — that is, if $A^c \in \mathcal{T}$.

Remarks

(i) If we wish to emphasize the topology $\mathcal{T}$, then we refer to A as $\mathcal{T}$-*closed*, or *closed for* $\mathcal{T}$ or *closed in* $(X, \mathcal{T})$.

(ii) We usually denote a closed set by D (or C), just as we usually denote an open set by U (or V).

Since $(A^c)^c = A$, for every set A, the following result is immediate.

Figure 1.3

Theorem 1.1

Let $(X, \mathcal{T})$ be a topological space. A set $A \subseteq X$ is open if and only if A^c is closed.

Problem 1.3

Let $X = \{a, b, c\}$ and let $\mathcal{T} = \{\varnothing, \{a\}, \{b, c\}, X\}$ be a topology on X.

(a) Which subsets of X are closed?

(b) Which subsets of X are both open and closed?

(c) Which subsets of X are neither open nor closed?

You may assume that $\mathcal{T}$ is a topology on X.

Thus a set may be open, closed, both or neither.

Problem 1.4

Let $(X, \mathcal{T})$ be a topological space. Show that $\varnothing$ and X are both closed.

Thus a non-empty topological space contains at least two sets that are both open and closed.

A set that is both open and closed is sometimes called a **clopen** set.

Problem 1.5

Let X be a set.

(a) Show that, if $\mathcal{T}$ is the discrete topology on X, then every subset of X is both open and closed.

(b) Show that, if $\mathcal{T}$ is the indiscrete topology on X, then the only closed sets are $\varnothing$ and X.

The following results are consequences of our definition of closed sets.

Theorem 1.2

Let (X, d) be a metric space, let $a \in X$ and let $r \geq 0$. The following are closed sets:

(a) the closed ball

$$B_d[a, r] = \{x \in X : d(a, x) \leq r\};$$

(b) the complement

$$B_d(a, r)^c = \{x \in X : d(a, x) \geq r\}$$

of the open ball $B_d(a, r)$;

(c) the sphere

$$S_d(a, r) = \{x \in X : d(a, x) = r\}.$$

Proof (a) follows from arguments similar to those used in Example 1.1, and (b) follows from the definition of a closed set. It remains only to prove (c). It is sufficient to observe that

$$S_d(a, r)^c = \mathbb{R}^2 - \{x : d(a, x) = r\}$$
$$= B_d(a, r) \cup \{x : d(a, x) > r\} = B_d(a, r) \cup (B_d[a, r])^c,$$

which is the union of two open sets and is therefore open. ∎

Remark

From (a), $\{a\} = B_d[a, 0]$ is closed, so that every one-point set in a metric space is closed.

1.2 Examples of closed sets

To gain familiarity with this new idea, we determine the closed sets for some of the topological spaces you met in *Unit A3*.

Discrete and indiscrete topologies

In Problem 1.5, we asked you to find the closed sets for the discrete and indiscrete topologies. The results are summarized as follows.

For the *discrete topology* on a set X:

every subset of X is both open and closed.

For the *indiscrete topology* on a set X:

$\varnothing$ and X are both open and closed; no other set is open or closed.

Deleted-point topology

Let X be a non-empty set, and fix $a \in X$. Recall that the *a-deleted-point topology* on X is

$$\mathcal{T}_a = \{X\} \cup \{U \subseteq X : a \notin U\}.$$

Worked problem 1.1

Determine which subsets of X are closed for the a-deleted-point topology on X.

Solution

Let $D \subseteq X$ be closed.

We know that D is closed if and only if its complement D^c is open — so, either D^c is X or D^c does not contain a.

This means that either $D = \varnothing$ or D contains a.

Hence, the collection of closed subsets of X is

$$\{\varnothing\} \cup \{D \subseteq X : a \in D\}.$$

∎

Problem 1.6

Determine which one-point subsets of X are closed for the a-deleted-point topology on X.

Co-finite topology

Let X be a set. Recall that the *co-finite topology* on X is

$$\mathcal{T} = \{\varnothing\} \cup \{U \subseteq X : U^c \text{ is finite}\}.$$

Problem 1.7

Determine which subsets of X are closed for the co-finite topology on X.

Subspace topology

Let $(X, \mathcal{T})$ be a topological space and let A be a subset of X. Recall that the *subspace topology* on A inherited from $\mathcal{T}$ is the family $\mathcal{T}_A = \{U \cap A : U \in \mathcal{T}\}$. We want to determine the $\mathcal{T}_A$-closed sets.

Since $\mathcal{T}_A$ is a topology on A (and not on X), we must take the complement of an open set $V \in \mathcal{T}_A$ with respect to A, and not to X. Hence, the closed sets for the subspace topology are the members of the family

$$\mathcal{F}_A = \{A - V : V \in \mathcal{T}_A\}.$$

Here, to avoid ambiguity, we use the notation $A - V$, since it would not be clear whether V^c denoted the complement with respect to A or X.

How can we describe these sets in terms of the open sets of our original topology $\mathcal{T}$? Now, by definition, V is in $\mathcal{T}_A$ if and only if there is an open set $U \in \mathcal{T}$ with $V = U \cap A$. So $C \in \mathcal{F}_A$ if and only if there is an open set $U \in \mathcal{T}$ such that

$$C = A - (U \cap A) = (X \cap A) - (U \cap A) = (X - U) \cap A.$$

Since $U \in \mathcal{T}$, its complement $U^c = X - U$ is closed for the original topology $\mathcal{T}$, and so we have the following result.

Figure 1.4

Theorem 1.3

Let $(A, \mathcal{T}_A)$ be a subspace of the topological space $(X, \mathcal{T})$. Then a set $C \subseteq A$ is $\mathcal{T}_A$-closed if and only if there is a $\mathcal{T}$-closed set D such that

$$C = D \cap A.$$
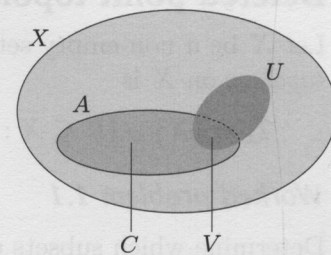
Problem 1.8

Let $(A, \mathcal{T}_A)$ be a subspace of the topological space $(X, \mathcal{T})$.

(a) Prove that if $D \subseteq A$ is $\mathcal{T}$-closed, then D is $\mathcal{T}_A$-closed.

(b) Give an example to show that it is not *always* true that if $C \subseteq A$ is $\mathcal{T}_A$-closed, then C is $\mathcal{T}$-closed.

Although, as Problem 1.8(b) shows, it is not always true that if $C \subseteq A$ is $\mathcal{T}_A$-closed then C is $\mathcal{T}$-closed, the result does hold when A is $\mathcal{T}$-closed.

You are asked to prove this in the exercises for this unit.

1.3 Properties of closed sets

Having looked at several examples of closed sets, we now investigate the *properties* of closed sets. In particular, we show that we can characterize the family of closed sets axiomatically, just as we did for open sets. We obtain this characterization from that for open sets by use of De Morgan's Laws, which you will recall state that, if $\{A_i : i \in I\}$ is a family of subsets of X, then

Unit A3, Theorem 2.10.

$$\left(\bigcup_{i \in I} A_i \right)^c = \bigcap_{i \in I} A_i^c \quad \text{and} \quad \left(\bigcap_{i \in I} A_i \right)^c = \bigcup_{i \in I} A_i^c.$$

In order to derive the axiomatic characterization of closed sets, we need the following notation.

Definition

Let $\mathcal{F}$ be a family of subsets of X. Then

$$\mathcal{F}^{(c)} = \{A^c : A \in \mathcal{F}\}.$$

This is purely notation: $\mathcal{F}^{(c)}$ is defined only in relation to $\mathcal{F}$; there is no assumption that the sets in $\mathcal{F}$ have any special topological properties.

For example, if $X = \mathbb{R}$ and $\mathcal{F} = \{[n, \infty) : n \in \mathbb{Z}\}$, then

$$\mathcal{F}^{(c)} = \{\mathbb{R} - [n, \infty) : n \in \mathbb{Z}\} = \{(-\infty, n) : n \in \mathbb{Z}\}.$$

Remark

If $\mathcal{T}$ is a topology on X, then $\mathcal{T}^{(c)}$ is the family of $\mathcal{T}$-closed sets.

Theorem 1.4

Let $(X, \mathcal{T})$ be a topological space. Then the collection $\mathcal{T}^{(c)}$ of closed sets satisfies the following three conditions:

(C1) the sets $\varnothing$ and X belong to $\mathcal{T}^{(c)}$;

(C2) the union of any *two* sets in $\mathcal{T}^{(c)}$ belongs to $\mathcal{T}^{(c)}$;

(C3) the intersection of *any* collection of sets in $\mathcal{T}^{(c)}$ belongs to $\mathcal{T}^{(c)}$.

Conversely, if $\mathcal{C}$ is a collection of subsets of X satisfying conditions (C1)–(C3), then $\mathcal{C}^{(c)}$ is a topology on X.

Remarks

(i) Note the contrast between these conditions and the three conditions (T1)–(T3) for collections of open sets. By De Morgan's Laws, complementation turns the *intersection* of two open sets into the *union* of two closed sets, and arbitrary *unions* of open sets into arbitrary *intersections* of closed sets.

(ii) If (C2) holds, then we may also conclude that the union of any *finite* collection of sets in $\mathcal{T}^{(c)}$ belongs to $\mathcal{T}^{(c)}$. The argument is analogous to that used for (T2) for open sets.   See *Unit A2*, Section 4.

Proof Let $\mathcal{T}$ be a topology on X.

We first show that $\mathcal{T}^{(c)}$ satisfies (C1)–(C3).

(C1) This is the result of Problem 1.4.

(C2) Let $D_1, D_2 \in \mathcal{T}^{(c)}$, and let $D = D_1 \cup D_2$. We show that $D \in \mathcal{T}^{(c)}$.
By De Morgan's First Law,
$$D^c = (D_1 \cup D_2)^c = D_1^c \cap D_2^c.$$
Thus D^c is the intersection of two open sets and so is open by (T2).
Hence $D = (D^c)^c \in \mathcal{T}^{(c)}$.
Thus (C2) is satisfied.

(C3) Let $\mathcal{F} = \{D_i : i \in I\}$ be a collection of sets in $\mathcal{T}^{(c)}$ and let $D = \bigcap_{i \in I} D_i$.
We show that $D \in \mathcal{T}^{(c)}$.
By De Morgan's Second Law,
$$D^c = \left(\bigcap_{i \in I} D_i \right)^c = \bigcup_{i \in I} D_i^c.$$
Since $D_i^c \in \mathcal{T}$, for each $i \in I$, it follows from (T3) that D^c is open and so $D = (D^c)^c \in \mathcal{T}^{(c)}$.
Thus (C3) is satisfied.

We ask you to prove the converse in the following problem.   ∎

Problem 1.9

Let X be a set and let $\mathcal{C}$ be a collection of subsets of X that satisfies (C1)–(C3). Prove that $\mathcal{C}^{(c)}$ satisfies (T1)–(T3) and is therefore a topology on X.

1.4 Continuity and closed sets

Given the symmetry between open and closed sets and the fact that a topology can be defined in terms of a family of sets satisfying (C1)–(C3), it should come as no surprise that we can define continuity in terms of closed sets. This alternative definition is an immediate consequence of the following result.

Theorem 1.5

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.

A function $f: X \to Y$ is $(\mathcal{T}_X, \mathcal{T}_Y)$-continuous if and only if $f^{-1}(D)$ is $\mathcal{T}_X$-closed whenever D is $\mathcal{T}_Y$-closed.

Proof We show that:

(a) if f is continuous, then $f^{-1}(D) \in \mathcal{T}_X^{(c)}$ whenever $D \in \mathcal{T}_Y^{(c)}$;

(b) if $f^{-1}(D) \in \mathcal{T}_X^{(c)}$ whenever $D \in \mathcal{T}_Y^{(c)}$, then f is continuous.

Proof of (a)

Let f be continuous and let $D \in \mathcal{T}_Y^{(c)}$. Then $D^c \in \mathcal{T}_Y$ and, since f is continuous, $f^{-1}(D^c) \in \mathcal{T}_X$. Thus $(f^{-1}(D^c))^c \in \mathcal{T}_X^{(c)}$. Now

$$(f^{-1}(A))^c = f^{-1}(A^c),$$

for any set $A \subseteq Y$ (as Figure 1.5 illustrates), and so $f^{-1}(D) = (f^{-1}(D^c))^c \in \mathcal{T}_X^{(c)}$, as required.

Proof of (b)

Suppose that $f^{-1}(D) \in \mathcal{T}_X^{(c)}$ whenever $D \in \mathcal{T}_Y^{(c)}$. We need to show that if $U \in \mathcal{T}_Y$ then $f^{-1}(U) \in \mathcal{T}_X$. So let $U \in \mathcal{T}_Y$. Then $U^c \in \mathcal{T}_Y^{(c)}$, and so $f^{-1}(U^c) \in \mathcal{T}_X^{(c)}$. Now

$$f^{-1}(U^c) = (f^{-1}(U))^c,$$

and so $f^{-1}(U) \in \mathcal{T}_X$. Therefore f is continuous. ∎

We thus have the following alternative definition of continuity for functions between topological spaces.

Figure 1.5

Definition

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.

A function $f: X \to Y$ is **continuous** if $f^{-1}(D) \in \mathcal{T}_X^{(c)}$ whenever $D \in \mathcal{T}_Y^{(c)}$.

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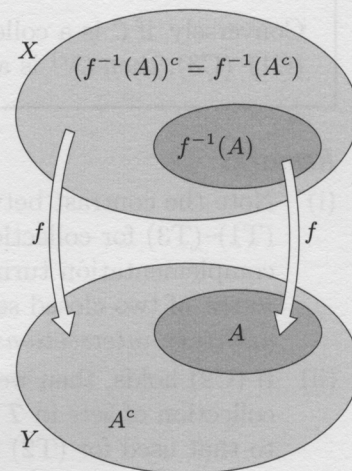
2 Closure

After working through this section, you should be able to:

▶ determine whether a given set is a *neighbourhood* of a given point;

▶ determine the *closure points* of a given set;

▶ state the connection between the closure points and the *closure* of a set;

▶ find the closure of a given set.

In this section, we consider the role that individual points play in determining whether a given set is open or closed. For example, for $(\mathbb{R}, \mathcal{T}(d^{(1)}))$, the interval $(0, 1)$ is open, the interval $[0, 1]$ is closed, and the interval $(0, 1]$ is neither open nor closed. Our objective is to determine which properties of these sets in the vicinity of the points 0 and 1 determine whether these sets are open or closed (or neither).

Once we have done this, we shall see how to associate a closed set with an arbitrary set in a natural way. This enables us (in Section 4) to investigate such notions as the *interior*, *exterior* and *boundary* of an arbitrary subset of a topological space.

2.1 Neighbourhoods

In order to investigate the structure of a set in a topological space in the vicinity of a given point, it is helpful to work with open sets containing the point. Topologists have given these sets a name.

Definition

Let $(X, \mathcal{T})$ be a topological space and let $x \in X$. A **neighbourhood** of x is an open set U that contains x.

Some books call these sets *open neighbourhoods*.

Remarks

(i) If we wish to emphasize the topology $\mathcal{T}$, then we refer to U as a *$\mathcal{T}$-neighbourhood of x*, or a *neighbourhood of x for $\mathcal{T}$* or a *neighbourhood of x in $(X, \mathcal{T})$*.

(ii) The whole space X is a neighbourhood of each point $x \in X$, so that every point has at least one neighbourhood.

(iii) If $a \in A \subseteq X$ and U is a neighbourhood of a that is contained in A, then we say that U is a *neighbourhood of a in A*.

(iv) If $\mathcal{B}$ is a base for $\mathcal{T}$ and U is open, then U can be written as a union of sets from $\mathcal{B}$. This implies that, for every neighbourhood U of a point x, there is a (necessarily open) base set $B \in \mathcal{B}$ with $x \in B \subseteq U$.

Bases were introduced in *Unit A3*, Section 5.

Problem 2.1 _____

Let $\mathcal{T}_a$ be the a-deleted-point topology on a set X.

(a) Describe the neighbourhoods of a.

(b) Let $x \in X$ with $x \neq a$. Describe the neighbourhoods of x.

_____

We can use the idea of a neighbourhood to provide a most useful characterization of open sets.

Theorem 2.1

Let $(X, \mathcal{T})$ be a topological space. A set $U \subseteq X$ is open if and only if it contains a neighbourhood of each of its points.

Proof If U is open, then it is a neighbourhood of each of its points.

Conversely, if for each point $x \in U$ there is a neighbourhood U_x of x that is contained in U, then

$$U = \bigcup_{x \in U} U_x.$$

This exhibits U as a union of open sets. Thus, by (T3), U is open.   ■
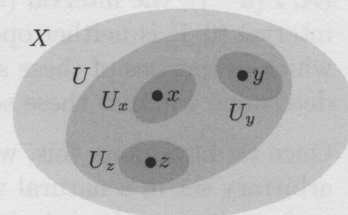
Figure 2.1

2.2 Closure points

Our definition of a closed set does not specify any property of the points of the set, but simply that its complement is open. We now determine a property of the points of a set that guarantees that it is closed. To deduce what such a property might be, let us look at some particular subsets of $\mathbb{R}$ (with the Euclidean topology).

We begin with the closed interval $A = [0, 1]$. If $a \in A$, then each neighbourhood U of a meets the set A (that is, it has a non-empty intersection with A). Moreover, if $x \notin A$, then, as A^c is open, there is a neighbourhood of x (for example A^c) that does not meet A.

Now consider the open interval $B = (0, 1)$. Again, every neighbourhood of every point of B meets B. Also, every neighbourhood of each of the points 0 and 1 meets B. In this case, the points 0 and 1 do not belong to B but still have the property that all of their neighbourhoods meet B.

Points with the property that every neighbourhood meets the set are called the *closure points* of the set. For the intervals A and B, the set of closure points is $[0, 1] = A$. The closed interval A contains all its closure points, whereas the open interval B does not.

Definition

Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. A point $x \in X$ is a **closure point** of A if each neighbourhood of x intersects A.

The **closure** of A is

$$\mathrm{Cl}(A) = \{x \in X : x \text{ is a closure point of } A\}.$$
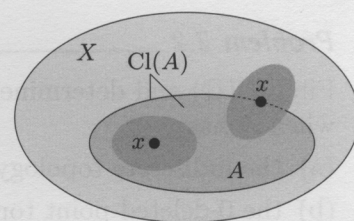
Figure 2.2

Remarks

(i) If we wish to emphasize the topology then we refer to x as a closure point *for* $\mathcal{T}$ or *in* $(X, \mathcal{T})$. Similarly, we may refer to the $\mathcal{T}$-*closure* of A, or to the closure of A *for* $\mathcal{T}$ or *in* $(X, \mathcal{T})$.

(ii) Clearly every point of A is a closure point of A. However, the converse is not true: a closure point of A may or may not belong to A. For example, 0 and 1 are closure points of the intervals $(0, 1)$ and $[0, 1]$.

(iii) For any topological space $(X, \mathcal{T})$, we can deduce immediately from the definition that $\mathrm{Cl}(\varnothing) = \varnothing$ and $\mathrm{Cl}(X) = X$.

The following simple properties of closures, which follow immediately from the definition, are very useful.

Theorem 2.2

Let $(X, \mathcal{T})$ be a topological space and let $A, B \subseteq X$ with $A \subseteq B$. Then

$$A \subseteq \mathrm{Cl}(A) \quad \text{and} \quad \mathrm{Cl}(A) \subseteq \mathrm{Cl}(B).$$

The following worked problem shows that changing the topology on a set can change the set of closure points.

Worked problem 2.1

(a) Show that, for the Euclidean topology on $\mathbb{R}$, $\mathrm{Cl}(\mathbb{Q}) = \mathrm{Cl}(\mathbb{Q}^c) = \mathbb{R}$.

(b) Find $\mathrm{Cl}(\mathbb{Q})$ for the discrete topology on $\mathbb{R}$.

Solution

(a) We begin by showing that $\mathrm{Cl}(\mathbb{Q}) = \mathbb{R}$. We know, by Theorem 2.2, that $\mathbb{Q} \subseteq \mathrm{Cl}(\mathbb{Q})$. So, since $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$, we need show only that the irrationals are closure points of $\mathbb{Q}$. Let $x \in \mathbb{Q}^c$ and suppose that U is a neighbourhood of x. Since U is open with respect to the Euclidean topology on $\mathbb{R}$, there is a real number $r > 0$ such that $(x - r, x + r) \subseteq U$. Now, between any two distinct real numbers there is a rational number and so $(x - r, x + r)$ intersects $\mathbb{Q}$. Thus each neighbourhood of x contains a point in $\mathbb{Q}$ and so $x \in \mathrm{Cl}(\mathbb{Q})$. Hence $\mathbb{Q}^c \subseteq \mathrm{Cl}(\mathbb{Q})$, as required.

Similar arguments can be used to show that $\mathrm{Cl}(\mathbb{Q}^c) = \mathbb{R}$,

(b) By Theorem 2.2, $\mathbb{Q} \subseteq \mathrm{Cl}(\mathbb{Q})$, so all the rational numbers are closure points. We need to determine whether the points of $\mathbb{Q}^c$ are closure points of $\mathbb{Q}$. Let $x \in \mathbb{Q}^c$. For the discrete topology, $\{x\}$ is a neighbourhood of x, and this neighbourhood does not meet $\mathbb{Q}$. Therefore, x is not a closure point of $\mathbb{Q}$. Hence, for this topology, no irrational number is a closure point of $\mathbb{Q}$. Therefore $\mathrm{Cl}(\mathbb{Q}) = \mathbb{Q}$. ∎

Problem 2.2

Find $\mathrm{Cl}(\mathbb{Q})$ and determine whether $\mathrm{Cl}(\mathbb{Q})$ is open, closed, neither or both, when $\mathbb{R}$ has:

(a) the indiscrete topology;

(b) the 0-deleted-point topology.

For each of the examples in Problem 2.2, the closure of $\mathbb{Q}$ is a closed set. Our next result shows that (as we might hope) the closure of a set is *always* a closed set.

Lemma 2.3

Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. Then $\mathrm{Cl}(A)$ is closed.

Proof In order to show that $\mathrm{Cl}(A)$ is closed, it is enough to show that $\mathrm{Cl}(A)^c$ is open. Let $x \in \mathrm{Cl}(A)^c = X - \mathrm{Cl}(A)$. Since x is not a closure point of A, there is a neighbourhood U_x of x that does not intersect A, i.e. $U_x \cap A = \varnothing$.

We claim that U_x does not meet $\mathrm{Cl}(A)$. We use proof by contradiction. Assume that there is a point $y \in U_x \cap \mathrm{Cl}(A)$ and let U_y be a neighbourhood of y. Now $U_y \cap U_x$ is also a neighbourhood of y and so, since $y \in \mathrm{Cl}(A)$, we have $(U_y \cap U_x) \cap A \neq \varnothing$. This implies that $U_x \cap A \neq \varnothing$, which is a contradiction. So $U_x \cap \mathrm{Cl}(A) = \varnothing$.

Therefore $U_x \subseteq \mathrm{Cl}(A)^c$, and hence $\mathrm{Cl}(A)^c$ contains a neighbourhood of each of its points. It follows from Theorem 2.1 that $\mathrm{Cl}(A)^c$ is open, as required. ■

We now prove that closed sets are precisely those sets that consist of their closure points.

Theorem 2.4

Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. Then A is closed if and only if $\mathrm{Cl}(A) = A$.

Proof We know by Lemma 2.3 that $\mathrm{Cl}(A)$ is a closed set. So, if $\mathrm{Cl}(A) = A$, then A is also a closed set.

Now suppose that A is closed. We must show that $\mathrm{Cl}(A) = A$. We know from Theorem 2.2 that $A \subseteq \mathrm{Cl}(A)$, and so it is enough to show that $\mathrm{Cl}(A) \subseteq A$. We do this by showing that $\mathrm{Cl}(A) \cap A^c = \varnothing$.

If $A^c = \varnothing$ then there is nothing to prove. So, let $A^c \neq \varnothing$ and let $x \in A^c$. Since A^c is open, it is a neighbourhood of x that does not meet A, and so x is not a closure point of A. Thus $\mathrm{Cl}(A) \cap A^c = \varnothing$, and so $\mathrm{Cl}(A) \subseteq A$. Thus, if A is closed, $\mathrm{Cl}(A) = A$. ■

Remarks

(i) A consequence of this theorem is that, if A is not a closed set, it does not contain all its closure points: A is a *proper* subset of $Cl(A)$.

(ii) Since $Cl(A)$ is a closed set, another consequence of the theorem is that

$$Cl(Cl(A)) = Cl(A).$$

(iii) Theorem 2.4, together with the result of Worked problem 2.1, tells us that $\mathbb{Q}$ is a closed subset of $\mathbb{R}$ for the discrete topology on $\mathbb{R}$.

> In fact, the argument used in the solution to Worked problem 2.1 can be used, together with Theorem 2.4, to show that *every subset of $\mathbb{R}$ is closed for the discrete topology.*

2.3 Properties of closures

Of all the closed sets containing a set A, $Cl(A)$ is quite special. The following theorem says that the closure of a set A is the smallest closed subset of X that contains A, that is, the **intersection** of all closed subsets that contain A.

Theorem 2.5

Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. Then

$$Cl(A) = \bigcap_{D \supseteq A,\, D \text{ closed}} D.$$

Proof Let $C = \bigcap_{D \supseteq A,\, D \text{ closed}} D$. We must show that $Cl(A) = C$.

We know from Theorem 2.2 and Lemma 2.3 that $Cl(A)$ is a closed set containing A, and so $C \subseteq Cl(A)$.

We now show that $Cl(A) \subseteq C$. Clearly $A \subseteq C$ and so, by Theorem 2.2, $Cl(A) \subseteq Cl(C)$. It follows from (C3) that C is closed and so, by Theorem 2.4, $Cl(C) = C$. Thus $Cl(A) \subseteq C$.

Therefore $Cl(A) = C$. ∎

> This proof illustrates an interesting technique: to find the smallest set with a certain property, take the intersection of all sets with that property. This works provided that the property in question is preserved under intersection.

Suppose that we have two subsets of a topological space. We have already observed that if $A \subseteq B$ then $Cl(A) \subseteq Cl(B)$. But, suppose we do not know whether one set is contained in the other. What is the relationship between the closure of their union and the union of their closures? What happens when we look at the intersection of their closures? The next theorem provides the answers.

> Theorem 2.2.

Theorem 2.6

Let $(X, \mathcal{T})$ be a topological space and let $A, B \subseteq X$. Then:
(a) $Cl(A \cup B) = Cl(A) \cup Cl(B)$;
(b) $Cl(A \cap B) \subseteq Cl(A) \cap Cl(B)$.

Proof We first prove (a).

We start by showing that

$$\mathrm{Cl}(A \cup B) \subseteq \mathrm{Cl}(A) \cup \mathrm{Cl}(B).$$

Since, by Theorem 2.2, $A \subseteq \mathrm{Cl}(A)$ and $B \subseteq \mathrm{Cl}(B)$, we have

$$A \cup B \subseteq \mathrm{Cl}(A) \cup \mathrm{Cl}(B),$$

and so, also by Theorem 2.2,

$$\mathrm{Cl}(A \cup B) \subseteq \mathrm{Cl}(\mathrm{Cl}(A) \cup \mathrm{Cl}(B)).$$

But, by Lemma 2.3, $\mathrm{Cl}(A)$ and $\mathrm{Cl}(B)$ are closed sets, and so, by (C2), $\mathrm{Cl}(A) \cup \mathrm{Cl}(B)$ is also a closed set. Hence, by Theorem 2.4,

$$\mathrm{Cl}(\mathrm{Cl}(A) \cup \mathrm{Cl}(B)) = \mathrm{Cl}(A) \cup \mathrm{Cl}(B)$$

Therefore

$$\mathrm{Cl}(A \cup B) \subseteq \mathrm{Cl}(A) \cup \mathrm{Cl}(B).$$

It remains to show that

$$\mathrm{Cl}(A) \cup \mathrm{Cl}(B) \subseteq \mathrm{Cl}(A \cup B).$$

To do this, notice that $A \subseteq A \cup B \subseteq \mathrm{Cl}(A \cup B)$ and $B \subseteq A \cup B \subseteq \mathrm{Cl}(A \cup B)$. Hence,

$$\mathrm{Cl}(A) \subseteq \mathrm{Cl}(\mathrm{Cl}(A \cup B)) = \mathrm{Cl}(A \cup B)$$

and

$$\mathrm{Cl}(B) \subseteq \mathrm{Cl}(\mathrm{Cl}(A \cup B)) = \mathrm{Cl}(A \cup B).$$

Therefore

$$\mathrm{Cl}(A) \cup \mathrm{Cl}(B) \subseteq \mathrm{Cl}(A \cup B).$$

Thus

$$\mathrm{Cl}(A) \cup \mathrm{Cl}(B) = \mathrm{Cl}(A \cup B)$$

as required.

We ask you to prove (b) in Problem 2.3. ∎

Problem 2.3 _____

Let $(X, \mathcal{T})$ be a topological space and let $A, B \subseteq X$. Prove that

$$\mathrm{Cl}(A \cap B) \subseteq \mathrm{Cl}(A) \cap \mathrm{Cl}(B).$$

Problem 2.4 _____

By considering $\mathbb{Q}$ and $\mathbb{Q}^c$ for the Euclidean topology on $\mathbb{R}$, show that $\mathrm{Cl}(A \cap B)$ may be a _proper_ subset of $\mathrm{Cl}(A) \cap \mathrm{Cl}(B)$.

2.4 Examples of closures

In this subsection, we determine the closures of various sets for our standard topological spaces.

Euclidean line

We begin by considering $\mathbb{R}$ with the Euclidean topology. We examine some representative subsets of $\mathbb{R}$ to find their closure points, and so determine their closures.

We use the term *Euclidean line* as a shorthand for $\mathbb{R}$ with the Euclidean topology.

We saw earlier that the closure points of $(0, 1)$ are the points of $(0, 1)$ together with 0 and 1; so the closure of $(0, 1)$ is $[0, 1]$. Similarly, if $a < b$, then the closures of (a, b), $(a, b]$, $[a, b)$ and $[a, b]$ are all $[a, b]$.

For a one-point set $\{a\}$, the only closure point is a itself, so that $\{a\}$ is closed for the Euclidean topology.

The closure points of the infinite interval (a, ∞) are the points of (a, ∞) together with the point a. Hence the closure of (a, ∞) is $[a, \infty)$. Similarly, the closure of $(-\infty, a)$ is $(-\infty, a]$.

Worked problem 2.2

Let $A = \{\frac{1}{n} : n \in \mathbb{N}\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$. Find $\mathrm{Cl}(A)$ for the Euclidean topology on $\mathbb{R}$.

Solution

By Theorem 2.2, the points of A belong to $\mathrm{Cl}(A)$, so it remains only to decide which points of A^c belong to $\mathrm{Cl}(A)$. Now $A \subseteq [0, 1]$, a closed set, and so, by Theorems 2.2 and 2.4,

$$\mathrm{Cl}(A) \subseteq \mathrm{Cl}([0, 1]) = [0, 1].$$

Hence we need only decide which points in $A^c \cap [0, 1]$ are closure points of A.

Suppose that $x \in A^c \cap [0, 1]$.

If $x \neq 0$, then there exists an $n \in \mathbb{N}$ for which

$$\frac{1}{n + 1} < x < \frac{1}{n}.$$

If we let $0 < r \leq \min\{x - \frac{1}{n+1}, \frac{1}{n} - x\}$, then $(x - r, x + r) \cap A = \varnothing$. Hence, we have found a neighbourhood $(x - r, x + r)$ of x that does not intersect A, and so x is not a closure point of A.

$(x - r, x + r)$

$\frac{1}{n+1} \quad\quad x \quad\quad \frac{1}{n}$

Figure 2.3

If $x = 0$, then, by the fried-egg property, any neighbourhood of x contains an interval $(0 - r, 0 + r) = (-r, r)$, for some $r > 0$. However small r is, there exists $n \in \mathbb{N}$ such that $\frac{1}{n} < r$, and so $(-r, r) \cap A \neq \varnothing$. Hence any neighbourhood of 0 intersects A. Thus 0 is a closure point of A.

Therefore $\mathrm{Cl}(A) = A \cup \{0\}$. ∎

Problem 2.5

Let (a_n) be a strictly increasing sequence of real numbers that is bounded above. If $A = \{a_n : n \in \mathbb{N}\}$, find $\mathrm{Cl}(A)$ for the Euclidean topology on $\mathbb{R}$.

Hint Recall that an increasing sequence of real numbers that is bounded above converges to a limit $a \in \mathbb{R}$.
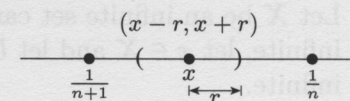
Recall that (a_n) is strictly increasing if $a_n < a_{n+1}$ for all $n \in \mathbb{N}$.

Indiscrete and discrete topologies

Let X be a non-empty set with the indiscrete topology. We saw in
Section 1 that the only non-empty closed subset is X, so (by Lemma 2.3)
the closure of any non-empty subset is X. This means that, for any
non-empty subset A, every point $x \in X$ is a closure point of A, whether or
not it belongs to A.

For the discrete topology, we saw in Section 1 that every set is closed, so
(by Theorem 2.4) the closure of any set is the set itself. Hence, the only
closure points of a set are the members of the set. Thus any point x not in
a set A cannot be a closure point of A.

Problem 2.6

Let X be a set with the discrete topology and let $x \in X$. Determine
$B_{d_0}[x, 1]$ and $\text{Cl}(B_{d_0}(x, 1))$.

Hint Recall from *Unit A2* that $B_{d_0}(x, 1) = \{x\}$.

Recall from *Unit A3* that the
discrete topology is equal to
$\mathcal{T}(d_0)$, where d_0 is the
discrete metric.

The solution to Problem 2.6 may surprise you, as you might have thought
that the closure of an open ball $B_d(x, r)$ is necessarily the closed ball
$B_d[x, r]$. This problem shows that this is not always the case.

Co-finite topology

Let X be an infinite set and let $\mathcal{T}$ be the co-finite topology on X. You saw
in Section 1 that a subset A of X is closed if and only if either $A = X$ or A
is finite.

Recall that, for the co-finite
topology, a non-empty set is
open if and only if its
complement is a finite set.

Finite subsets

If $A \subset X$ is finite, then A is closed, and so (by Theorem 2.4) $\text{Cl}(A) = A$.

Infinite subsets

If $A \subseteq X$ is infinite, then the only closed set containing A is X, and so (by
Theorem 2.2 and Lemma 2.3) $\text{Cl}(A) = X$.

Problem 2.7

Let X be an infinite set carrying the co-finite topology, let $A \subseteq X$ be
infinite, let $x \in X$ and let U be a neighbourhood of x. Show that $U \cap A$ is
infinite.

Continuous functions on $[0, 1]$

Recall that $C[0, 1]$ denotes the set of continuous functions on $[0,1]$ and that

$$d_{\max}(f, g) = \max\{|g(x) - f(x)| : x \in [0, 1]\}.$$

(You met the max metric in *Unit A2*, Subsection 2.3.)

Worked problem 2.3

Find the closure of

$$A = \{f \in C[0, 1] : f(\tfrac{1}{2}) = 0\}$$

for $\mathcal{T}(d_{\max})$.

$f_1 \in A$

$f_2 \notin A$
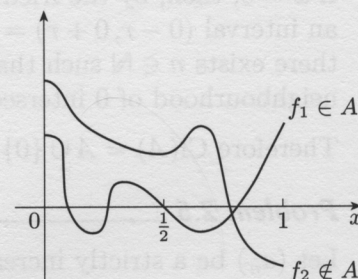
Figure 2.4

Solution

We know from Theorem 2.2 that $A \subseteq \mathrm{Cl}(A)$ and so it remains to determine which points $f \in A^c$ belong to $\mathrm{Cl}(A)$.

Let $f \in C[0,1]$ with $f \in A^c$. Then $f(\frac{1}{2}) \neq 0$. We use this to show that there are neighbourhoods of f that do not meet A, so that f is not a closure point of A. In fact, we shall find an open ball centred at f that contains no members of A.

Let $0 < r < |f(\frac{1}{2})|$ and consider the open ball

$$B_{d_{\max}}(f, r) = \{ g \in C[0,1] : d_{\max}(f, g) = \max_{x \in [0,1]} |g(x) - f(\dot{x})| < r \}.$$

Now assume that there is a function $g \in B_{d_{\max}}(f, r)$ which also belongs to A. Such a function satisfies $g(\frac{1}{2}) = 0$, and so

$$r < |f(\tfrac{1}{2})| = |f(\tfrac{1}{2}) - g(\tfrac{1}{2})| \leq d_{\max}(f, g) < r,$$

which is impossible. So $B_{d_{\max}}(f, r) \cap A = \varnothing$, and hence f is not a closure point of A.

Therefore $\mathrm{Cl}(A) = A$. ∎

Note that the value $\frac{1}{2}$ is in no way special: we could use any point $a \in [0,1]$. Thus, for each $a \in [0,1]$, the set

$$\{ f \in C[0,1] : f(a) = 0 \}$$

is closed. Correspondingly, for each $a \in [0,1]$ the complementary set

$$\{ f \in C[0,1] : f(a) \neq 0 \}$$

is open.

In the next worked problem we extend this result.

Worked problem 2.4

Find the closure of

$$A = \{ f \in C[0,1] : f(\tfrac{1}{n}) = 0 \text{ for each } n \in \mathbb{N} \}$$

for $\mathcal{T}(d_{\max})$.

Solution

We can write A as an *intersection* of closed sets:

$$A = \bigcap_{n=1}^{\infty} \{ f \in C[0,1] : f(\tfrac{1}{n}) = 0 \}.$$

It follows from (C3) that A is closed, and so, by Theorem 2.4, $\mathrm{Cl}(A) = A$. ∎

Problem 2.8

Let $I \subseteq [0,1]$ be a non-empty set. Find the closure of

$$A = \{ f \in C[0,1] : f(x) = 0 \text{ for } x \in I \}$$
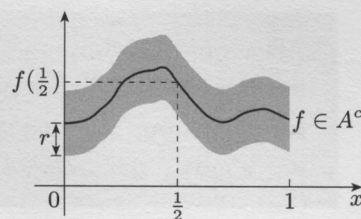
for $\mathcal{T}(d_{\max})$.

Figure 2.5

3 Dense sets and isolated points

After working through this section, you should be able to:
- ▶ explain the distinction between *dense* and *nowhere dense*;
- ▶ determine whether a given set is dense;
- ▶ identify the *isolated points* of a given set.

In this short section, we investigate two contrasting types of behaviour for a subset of the underlying set X of a topological space: a subset may be *dense* in X, in that each point in X has a point from the subset 'near' to it. In contrast, it may contain points that seem to be *isolated* from the rest of the subset.

3.1 Dense sets

In Worked problem 2.1 we used the fact that there is a rational number between any two distinct real numbers — i.e. that the rational numbers are *dense* in $\mathbb{R}$ — to help us show that, if $\mathbb{R}$ carries its Euclidean topology, then $\mathrm{Cl}(\mathbb{Q}) = \mathbb{R}$. But a subset does not have to be as large as $\mathbb{Q}$ for this result to be true: it depends on the topology.

For example, if $\mathbb{R}$ has its indiscrete topology, then the closure of *every* non-empty subset of $\mathbb{R}$ is equal to $\mathbb{R}$. In particular, even a very small subset of $\mathbb{R}$, such as $\{0\}$, has closure equal to $\mathbb{R}$. However, if $\mathbb{R}$ carries its discrete topology, then the closure of every set is equal to the set itself. In particular, even a very large set, such as $\mathbb{R} - \mathbb{Q}$, has closure not equal to $\mathbb{R}$. In fact, for the discrete topology, the only subset of $\mathbb{R}$ whose closure is $\mathbb{R}$ is $\mathbb{R}$ itself.

The closure of subsets for the indiscrete or discrete topology was discussed in Subsection 2.4.

A subset of $\mathbb{R}$ whose closure is equal to the whole of $\mathbb{R}$ is a *dense* subset of $\mathbb{R}$. More generally, we have the following definition.

***

Definitions

Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$.

The set A is **dense** in X if $\mathrm{Cl}(A) = X$.

The set A is **nowhere dense** in X if, whenever U is a non-empty open set of X , there is a non-empty open set $V \subseteq U$ for which $V \cap A = \varnothing$.
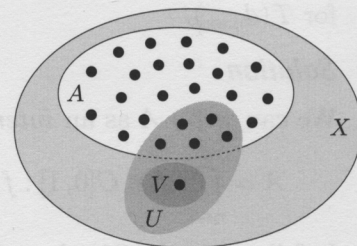
Figure 3.1   A nowhere dense set.

***

Remarks

(i) If we wish to emphasize the topology, then we refer to A as $\mathcal{T}$-*dense*, or dense *for* $\mathcal{T}$ or dense *in* $(X, \mathcal{T})$.

(ii) Since $\mathrm{Cl}(X) = X$, the underlying set X of a topological space is always dense in itself.

(iii) For any topological space, the empty set is nowhere dense.

(iv) A set A can be neither dense nor nowhere dense.

Worked problem 3.1

Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$ be dense in X. Show that if $A \subseteq B$ then B is also dense in X.

Solution

If $A \subseteq B$, then, by Theorem 2.2, $\mathrm{Cl}(A) \subseteq \mathrm{Cl}(B)$. But A is dense in X, so that $\mathrm{Cl}(A) = X$. Hence $\mathrm{Cl}(B) = X$: that is, B is dense in X. ∎

Problem 3.1

Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$ be nowhere dense in X. Show that if $B \subseteq A$, then B is nowhere dense in X.

Worked problem 3.2

Let X be a set with at least two elements. Let a be a point of X and let $\mathcal{T}_a$ be the a-deleted-point topology on X. Prove that $\{a\}^c$ is the only proper subset of X that is dense in X.

Solution

First we show that $\{a\}^c$ is dense. As $\{a\}^c$ differs from X only by the point a, we need show only that a is a closure point of $\{a\}^c$. Since

$$\mathcal{T}_a = \{X\} \cup \{U \subseteq X : a \notin U\},$$

the point a has only one neighbourhood: X itself. Since X intersects $\{a\}^c$, a is a closure point of $\{a\}^c$, and so $\{a\}^c$ is dense in X.

We must also show that no other proper subset A of X is dense — that is, if $A \neq X$ and $A \neq \{a\}^c$, then there is at least one point $x \in A^c$ that is not a closure point of A. Since $A \neq \{a\}^c$ and $A \neq X$, we can take $x \in A^c$ with $x \neq a$. But then $\{x\}$ is open, and hence is a neighbourhood of x that does not intersect A. Thus A is not dense. ∎

Problem 3.2

Let X be an infinite set carrying the co-finite topology. Show that:
(a) every infinite subset of X is dense in X;
(b) every finite subset of X is nowhere dense in X.

Problem 3.3

Let $(X, \mathcal{T})$ be a topological space and let $U, V \subseteq X$ be open dense subsets of X. Show that $U \cap V$ is also an open dense subset of X.

3.2 Isolated points

Consider the subset $A = [0, 1) \cup \{2\}$ of the Euclidean line. The point 2 is rather different in character from the points of $[0, 1)$ and from the point 1. In everyday language we say that 2 is *isolated*. The striking thing about 2 is that it is a closure point of A (as it belongs to A), yet it has neighbourhoods that contain no points of A other than 2 itself.
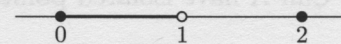

Figure 3.2

We are not saying that 2 has *no* neighbourhoods that meet $[0, 1)$. It does: $(0, 4)$ is one.

Definitions

Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$.

A point $a \in A$ is an **isolated point** of A if there is a neighbourhood U of a such that $U \cap A = \{a\}$.

Such a neighbourhood is called an **isolating neighbourhood**.
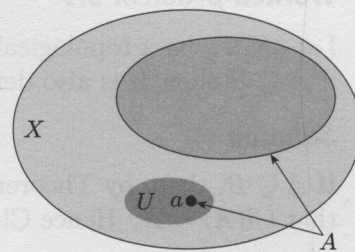
Figure 3.3

Remarks

(i) If we wish to emphasize the topology, then we refer to a as a $\mathcal{T}$-*isolated* point, or an *isolated point for* $\mathcal{T}$ or an *isolated point in* $(X, \mathcal{T})$. Similarly, we may refer to the isolating neighbourhood of a *for* $\mathcal{T}$ or *in* $(X, \mathcal{T})$.

(ii) An isolated point of a set A is necessarily an element of the set.

Worked problem 3.3

Identify the isolated points of

$$A = \{0\} \cup \{1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots\} = \{0\} \cup \bigcup_{n \in \mathbb{N}} \{\tfrac{1}{n}\}$$

for the Euclidean topology on $\mathbb{R}$.

Solution

Let $n \in \mathbb{N}$. Then $\frac{1}{n}$ is an isolated point of A since

$$(\tfrac{1}{n} - r, \tfrac{1}{n} + r) \cap A = \{\tfrac{1}{n}\},$$

for $0 < r \le \frac{1}{n} - \frac{1}{n+1}$.

Figure 3.4

It remains only to decide whether $\{0\}$ is an isolated point of A. But we have already shown in Worked problem 2.2 that every neighbourhood of 0 intersects the set $\{1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots\}$. Hence, 0 is not an isolated point of A. ∎
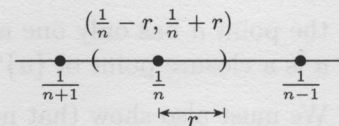
Problem 3.4

Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. Show that, if $a \in A$ and $\{a\}$ is an open set in the subspace topology on A, then a is an isolated point of A.

Let us now look at the existence, or otherwise, of isolated points for some of the topologies that we have met.

Indiscrete and discrete topologies

Let X be a non-empty set and suppose that $A \subseteq X$ has at least two points. Can A have isolated points for the indiscrete or discrete topology on X?

Worked problem 3.4

Show that if X has the indiscrete topology and $A \subseteq X$ contains at least two points, then A has no isolated points.

Solution

Let $a \in A$. We must show that every neighbourhood of a intersects A in more than just the point a. Let U be a neighbourhood of a. Then $U \neq \varnothing$, since it must contain a, and so $U = X$, since this is the only non-empty open set. But then $U \cap A = A$, which is not $\{a\}$, since A contains at least two points. Thus a is not an isolated point. ∎

Problem 3.5

Show that if X has the discrete topology and $A \subseteq X$, then every point in A is an isolated point.

Euclidean line

We now investigate whether certain subsets of the Euclidean line have isolated points.

Worked problem 3.5

Show that $\mathbb{Q}$ has no isolated points for the Euclidean topology on $\mathbb{R}$.

Solution

Let $x \in \mathbb{Q}$ and let U be a neighbourhood of x. Then there is a rational number $r > 0$ such that $(x - r, x + r) \subseteq U$. But the point $x + \frac{1}{2}r$ is rational and lies in $(x - r, x + r)$, and so $\mathbb{Q} \cap U \neq \{x\}$. Since this holds for all neighbourhoods of x, we conclude that x is not isolated, and so $\mathbb{Q}$ has no isolated points. ∎
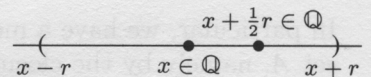
Figure 3.5

Problem 3.6

Show that $\mathbb{Q}^c$ has no isolated points for the Euclidean topology on $\mathbb{R}$.

Co-finite topology

Let X be an infinite set and let $\mathcal{T}$ be the co-finite topology on X. Recall from Subsection 2.4 that if $A \subseteq X$ is finite then it is closed, and if A is infinite then its closure is X. We now investigate the existence of isolated points for subsets of X.

Finite subsets

Suppose that A is finite and let $a \in A$. Then $A - \{a\}$ is finite, and so is closed. Its complement is thus a neighbourhood of a. Moreover, this neighbourhood contains no points of A other than a, and so a is isolated. Thus every point in a finite subset is isolated.

Infinite subsets

Suppose that A is infinite and let $a \in A$. Let U be a neighbourhood of a, so $a \in U$ and U^c is finite. Now $A - \{a\}$ is an infinite set, and so $A - \{a\} \nsubseteq U^c$. Hence

$$(A - \{a\}) \cap U \neq \varnothing,$$

and so $\{a\}$ is not an isolated point of A. Thus an infinite subset A contains no isolated points.

4 Interiors, exteriors and boundaries

After working through this section, you should be able to:

▶ define the *interior*, *exterior* and *boundary* of a set;

▶ find the interior, exterior and boundary of a given set;

▶ describe the behaviour of interiors, exteriors and boundaries under unions, intersections, complements and homeomorphisms.

Let us take stock of where we are. Suppose that $(X, \mathcal{T})$ is a topological space. We began the unit by defining a *closed set* as the complement of an open set. We then defined the *closure points* of a set $A \subseteq X$ and showed that they constitute the *closure* of A, the smallest closed set containing A. Moreover, A is *dense* if and only if $\mathrm{Cl}(A) = X$. An *isolated point* of a set A is a point $a \in A$ which has a neighbourhood intersecting A only at the point a.

In particular, we have a means of associating a closed set with an arbitrary set A, namely by the closure of A. Is there a natural way of associating an open set with A? In this section we show that there is. In doing so, we find a way to classify the points of our space X with respect to A so that we can say which points of X lie on the *inside*, on the *outside* and on the *boundary* of A.

4.1 Interior and exterior

We call the set of points 'inside' a set A the *interior* of A and the set of points 'outside' A the *exterior* of A. Before giving the formal definitions of these terms, we look at an example. Consider the interval $I = [0, 1)$. If $0 < x < 1$, then x is inside the interval I, and if $x < 0$ or $x > 1$, then x is outside the interval I. The points 0 and 1 constitute the boundary of I.

The key observation is that, if x is inside $[0, 1)$, then there is a neighbourhood of x that also lies inside $[0, 1)$ — for example, $(0, 1)$. Similarly, if x is outside $[0, 1)$, then there is a neighbourhood of x that also lies outside $[0, 1)$ — for example, $(-\infty, 0) \cup (1, \infty)$. This observation motivates the following definitions.

Definitions

Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$.

A point $x \in X$ is an **interior point** of A if there is a neighbourhood U of x with $U \subseteq A$.

The **interior** of A is the set

$$\mathrm{Int}(A) = \{x \in X : x \text{ is an interior point of } A\}.$$

A point $x \in X$ is an **exterior point** of A if there is a neighbourhood U of x that is disjoint from A, i.e. $U \cap A = \varnothing$. Equivalently, $U \subseteq A^c$.

The **exterior** of A is the set

$$\mathrm{Ext}(A) = \{x \in X : x \text{ is an exterior point of } A\}.$$

Remarks

(i) If we wish to emphasize the topology, then we refer to x as an interior point or exterior point of A *for* $\mathcal{T}$ or *in* $(X, \mathcal{T})$. Similarly, we may refer to the $\mathcal{T}$-*interior* or $\mathcal{T}$-*exterior* of A, or to the interior or exterior of A *for* $\mathcal{T}$ or *in* $(X, \mathcal{T})$.

(ii) Certainly, $\mathrm{Int}(A) \subseteq A$ and $\mathrm{Ext}(A) \subseteq A^c$. So points in the interior lie inside A and points in the exterior lie outside A. Also, $\mathrm{Int}(A) \cap \mathrm{Ext}(A) = \varnothing$.

(iii) For any topological space $(X, \mathcal{T})$, we can deduce immediately from the definition that $\mathrm{Int}(\varnothing) = \varnothing$, $\mathrm{Ext}(\varnothing) = X$, $\mathrm{Int}(X) = X$, $\mathrm{Ext}(X) = \varnothing$.

The following result is very useful.

Lemma 4.1

Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. Then

$$\mathrm{Int}(A) = \mathrm{Ext}(A^c) \quad \text{and} \quad \mathrm{Ext}(A) = \mathrm{Int}(A^c).$$

Proof Let $x \in X$.

If x is an interior point of A, then there is a neighbourhood U of x with $U \subseteq A$. Hence $U \cap A^c = \varnothing$, so that x is an exterior point of A^c. Thus, $\mathrm{Int}(A) \subseteq \mathrm{Ext}(A^c)$.

Conversely, if x is an exterior point of A^c, then there is a neighbourhood U of x with $U \cap A^c = \varnothing$. Hence $U \subseteq A$, so that x is an interior point of A^c. Thus, $\mathrm{Ext}(A^c) \subseteq \mathrm{Int}(A)$.

Therefore

$$\mathrm{Int}(A) = \mathrm{Ext}(A^c).$$

Repeating the above arguments with the roles of A and A^c interchanged, we obtain

$$\mathrm{Ext}(A) = \mathrm{Int}(A^c). \qquad \blacksquare$$

We now check that our intuition of what the interior and exterior of $[0, 1)$ should be agrees with the formal definition.

Worked problem 4.1

Find the interior and exterior of $[0, 1)$ for the Euclidean topology on $\mathbb{R}$.

Solution

Interior

Since $\mathrm{Int}([0, 1)) \subseteq [0, 1)$, we need only decide which points of $[0, 1)$ are interior points. If $x \in (0, 1)$, then $(0, 1)$ is a neighbourhood of x that lies entirely within $[0, 1)$. Hence every point of $(0, 1)$ is an interior point of $[0, 1)$. For $x = 0$, each neighbourhood of x contains points that are negative, and so not in $(0, 1)$. Hence 0 is not an interior point of $[0, 1)$. Thus

$$\mathrm{Int}([0, 1)) = (0, 1).$$

Exterior

We use the fact, from Lemma 4.1, that $\mathrm{Ext}(A) = \mathrm{Int}(A^c)$. Now

$$[0, 1)^c = (-\infty, 0) \cup [1, \infty),$$

and so the task is to find the interior points of $B = (-\infty, 0) \cup [1, \infty)$. If $x \in (-\infty, 0) \cup (1, \infty)$, then $(-\infty, 0) \cup (1, \infty)$ is a neighbourhood of x that is contained in B and so x is an interior point of B. If $x = 1$, then each neighbourhood of x has non-empty intersection with $[0, 1) = B^c$, and so x is not an interior point of B. Thus

$$\text{Ext}([0, 1)) = \text{Int}((-\infty, 0) \cup [1, \infty)) = (-\infty, 0) \cup (1, \infty).$$ ∎

Problem 4.1

Find the interior and exterior of $(0, 1) \cup \{2\}$ for the Euclidean topology on $\mathbb{R}$.

You may have noticed that the interiors and exteriors found in Worked problem 4.1 and Problem 4.1 are all open sets. This is no accident, as we now show.

Theorem 4.2

Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. Then

$$\text{Int}(A) = \bigcup_{U \subseteq A,\, U \text{ open}} U \quad \text{and} \quad \text{Ext}(A) = \bigcup_{U \subseteq A^c,\, U \text{ open}} U.$$

In particular, both $\text{Int}(A)$ and $\text{Ext}(A)$ are open sets.

Remark

This theorem states that the interior of a set $A \subseteq X$ is the largest open subset of X that is contained in A, and the exterior of a set $A \subseteq X$ is the largest open subset of X that is disjoint from A.

Proof We first show that $\text{Int}(A) \subseteq \bigcup_{U \subseteq A,\, U \text{ open}} U$.

Let $x \in \text{Int}(A)$. Then there is a neighbourhood U of x such that $U \subseteq A$. In particular, U is an open subset of A and so $x \in U \subseteq \bigcup_{U \subseteq A,\, U \text{ open}} U$. Hence $\text{Int}(A) \subseteq \bigcup_{U \subseteq A,\, U \text{ open}} U$.

We now show that $\bigcup_{U \subseteq A,\, U \text{ open}} U \subseteq \text{Int}(A)$.

Let $x \in \bigcup_{U \subseteq A,\, U \text{ open}} U$. Then, by the definition of this set, there is an open set U such $x \in U \subseteq A$. But U is a neighbourhood of x, and so x is an interior point of A. Hence $\bigcup_{U \subseteq A,\, U \text{ open}} U \subseteq \text{Int}(A)$.

Therefore

$$\bigcup_{U \subseteq A,\, U \text{ open}} U = \text{Int}(A).$$

Since $\text{Int}(A)$ can be written as a union of open sets, it is open, by (T3).

Since $A^c \subseteq X$, it follows from the result we have just proved and Lemma 4.1 that

$$\text{Ext}(A) = \text{Int}(A^c) = \bigcup_{U \subseteq A^c,\, U \text{ open}} U.$$

$\text{Ext}(A)$ is also open, by (T3). ∎

There are many situations such as this where Lemma 4.1 can be used to save a lot of work.

The next result follows from Theorem 4.2. We show that open sets are those sets that are equal to their interior and closed sets are those sets whose complement is equal to their exterior.

> ## Corollary 4.3
>
> Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. Then:
> (a) A is open if and only if $\text{Int}(A) = A$;
> (b) A is closed if and only if $\text{Ext}(A) = A^c$.

Proof

(a) From Theorem 4.2, we know that $\text{Int}(A)$ is open and so, if $\text{Int}(A) = A$, then A is also open.

   If A is open, then it follows from Theorem 4.2 that $A \subseteq \text{Int}(A)$. Since we always have $\text{Int}(A) \subseteq A$, it follows that $\text{Int}(A) = A$.

(b) Since $A^c \subseteq X$, it follows from (a) and Lemma 4.1 that A^c is open (that is, A is closed) if and only if $\text{Ext}(A) = \text{Int}(A^c) = A^c$. ∎

Problem 4.2

Let $(X, \mathcal{T})$ be a topological space. Use Corollary 4.3 to find the interior and exterior of $\varnothing$ and X.

The solution to this problem confirms Remark (iii) at the start of this subsection.

By Theorem 4.2, the interior and exterior of a set A are open sets and so their complements must be closed sets. We now give a simple characterization of $\text{Int}(A)$ and $\text{Ext}(A)$ in terms of closed sets associated with A. The characterization is useful for calculating the interiors and exteriors of sets.

> ## Corollary 4.4
>
> Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. Then
> $$\text{Int}(A) = \text{Cl}(A^c)^c \quad \text{and} \quad \text{Ext}(A) = \text{Cl}(A)^c.$$

Proof We first show that $\text{Cl}(A^c)^c \subseteq \text{Int}(A)$.

By Theorem 2.2, $A^c \subseteq \text{Cl}(A^c)$. It follows, on taking complements, that

$$\text{Cl}(A^c)^c \subseteq (A^c)^c = A.$$

But $\text{Cl}(A^c)^c$ is an open set, and so, by Theorem 4.2,

$$\text{Cl}(A^c)^c \subseteq \text{Int}(A).$$

We now show that $\text{Int}(A) \subseteq \text{Cl}(A^c)^c$.

Since $\text{Int}(A) \subseteq A$, it follows that

$$\text{Int}(A)^c \supseteq A^c.$$

Also, since, by Theorem 4.2, $\text{Int}(A)$ is open, $\text{Int}(A)^c$ is closed.
Hence, by Theorems 2.4 and 2.2,

$$\text{Int}(A)^c = \text{Cl}(\text{Int}(A)^c) \supseteq \text{Cl}(A^c).$$

So, on taking complements, we have

$$\text{Int}(A) \subseteq \text{Cl}(A^c)^c.$$

Therefore

$$\text{Int}(A) = \text{Cl}(A^c)^c.$$

The result for $\text{Ext}(A)$ follows immediately from the result we have just proved, together with the identity $\text{Ext}(A) = \text{Int}(A^c)$, from Lemma 4.1. $\blacksquare$

Worked problem 4.2

Find the interior and exterior of $[0,1]$ for the Euclidean topology on $\mathbb{R}$.

Solution

By Corollary 4.4,

$$\text{Int}([0,1]) = \text{Cl}([0,1]^c)^c = \text{Cl}((-\infty,0) \cup (1,\infty))^c.$$

But $\text{Cl}((-\infty,0) \cup (1,\infty)) = (-\infty,0] \cup [1,\infty)$ and so

See Subsection 2.4.

$$\text{Int}([0,1]) = ((-\infty,0] \cup [1,\infty))^c = (0,1).$$

Since $[0,1]$ is closed, it follows from Corollary 4.3 that

$$\text{Ext}([0,1]) = [0,1]^c = (-\infty,0) \cup (1,\infty).$$

$\blacksquare$

Problem 4.3

Use the result of Worked problem 2.2 to find the exterior of

$$A = \{1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots\} = \{\tfrac{1}{n} : n \in \mathbb{N}\}$$

for the Euclidean topology on $\mathbb{R}$.

4.2 Boundary

We have now defined the interior and the exterior of a set A, and have shown that both are open sets. We have also seen that there may be points of X that are not included in either the interior or exterior of A. For example, in $\mathbb{R}$ with the Euclidean topology, the interior of $[0,1]$ is $(0,1)$ and the exterior is $(-\infty,0) \cup (1,\infty)$, and so the points 0 and 1 are 'left over'; they seem to be acting as *boundary points*. This observation motivates the following definitions.

Definitions

Let $(X,\mathcal{T})$ be a topological space and let $A \subseteq X$.

A point $x \in X$ is a **boundary point** of A if each neighbourhood of x intersects both A and A^c.

The **boundary** of A is the set

$$\text{Bd}(A) = \{x \in X : x \text{ is a boundary point of } A\}.$$

Remarks

(i) If we wish to emphasize the topology, then we refer to x as a boundary point of A *for* $\mathcal{T}$ or *in* $(X,\mathcal{T})$. Similarly we may refer to the $\mathcal{T}$-*boundary* of A, or to the boundary of A *for* $\mathcal{T}$ or *in* $(X,\mathcal{T})$.

(ii) For any topological space $(X,\mathcal{T})$, we can deduce immediately from its definition that $\text{Bd}(\varnothing) = \varnothing$ and $\text{Bd}(X) = \varnothing$.

Theorem 4.5

Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. Then
$$\mathrm{Bd}(A) = (\mathrm{Ext}(A) \cup \mathrm{Int}(A))^c.$$

Proof We first show that $\mathrm{Bd}(A) \subseteq (\mathrm{Ext}(A) \cup \mathrm{Int}(A))^c$.

Let $x \in \mathrm{Bd}(A)$. Then x is not an exterior point since every neighbourhood of it intersects A, and x is not an interior point since every neighbourhood of it intersects A^c. So
$$x \in \mathrm{Ext}(A)^c \cap \mathrm{Int}(A)^c = (\mathrm{Ext}(A) \cup \mathrm{Int}(A))^c,$$
by De Morgan's First Law. Thus
$$\mathrm{Bd}(A) \subseteq (\mathrm{Ext}(A) \cup \mathrm{Int}(A))^c.$$

It remains to verify that $(\mathrm{Ext}(A) \cup \mathrm{Int}(A))^c \subseteq \mathrm{Bd}(A)$.

Let $x \in (\mathrm{Ext}(A) \cup \mathrm{Int}(A))^c$. Then x is not an interior point of A, and so each neighbourhood of x intersects A^c. Also, x is not an exterior point of A, and so each neighbourhood of x intersects A. Hence each neighbourhood of x intersects both A and A^c. So $x \in \mathrm{Bd}(A)$. Thus
$$(\mathrm{Ext}(A) \cup \mathrm{Int}(A))^c \subseteq \mathrm{Bd}(A).$$

Therefore
$$\mathrm{Bd}(A) = (\mathrm{Ext}(A) \cup \mathrm{Int}(A))^c. \qquad \blacksquare$$

Remarks

(i)   Theorem 4.5 tells us immediately that $\mathrm{Bd}(A) \cup \mathrm{Ext}(A) \cup \mathrm{Int}(A) = X$.
(ii) We can deduce from Theorem 4.5 that $\mathrm{Bd}(A) \cap \mathrm{Ext}(A) = \varnothing$ and $\mathrm{Bd}(A) \cap \mathrm{Int}(A) = \varnothing$. We saw earlier that $\mathrm{Int}(A) \cap \mathrm{Ext}(A) = \varnothing$. Thus $\mathrm{Bd}(A)$, Ext (A) and $\mathrm{Int}(A)$ are mutually disjoint.

The relationships between the interior, exterior and boundary of a set are illustrated in Figure 4.1. Note that $\mathrm{Bd}(A)$ is a closed set, disjoint from $\mathrm{Int}(A)$ and $\mathrm{Ext}(A)$. It is difficult to do justice to this in a drawing.
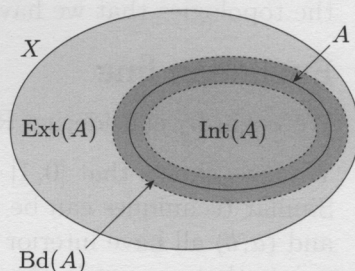
Figure 4.1

Problem 4.4

Let $\mathbb{R}$ have its Euclidean topology. Find the interior, exterior and boundary of (a) $\mathbb{Q}$ and (b) $\mathbb{Q}^c$.

Hint Use the result of Worked problem 2.1(a).

Corollary 4.4 gives a description of the interior and exterior of a set A in terms of closed sets associated with A. By using this result, we can obtain a similar result for the boundary of A.

Corollary 4.6

Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. Then
$$\mathrm{Bd}(A) = \mathrm{Cl}(A) \cap \mathrm{Cl}(A^c).$$

In particular, $\mathrm{Bd}(A)$ is a closed set.

Proof From Theorem 4.5, we know that

$$\mathrm{Bd}(A) = (\mathrm{Ext}(A) \cup \mathrm{Int}(A))^c.$$

But, by Corollary 4.4,

$$\mathrm{Int}(A) = \mathrm{Cl}(A^c)^c \quad \text{and} \quad \mathrm{Ext}(A) = \mathrm{Cl}(A)^c.$$

Hence, using De Morgan's First Law,

$$\begin{aligned}
\mathrm{Bd}(A) &= (\mathrm{Ext}(A) \cup \mathrm{Int}(A))^c \\
&= \mathrm{Ext}(A)^c \cap \mathrm{Int}(A)^c \\
&= (\mathrm{Cl}(A)^c)^c \cap (\mathrm{Cl}(A^c)^c)^c \\
&= \mathrm{Cl}(A) \cap \mathrm{Cl}(A^c).
\end{aligned}$$

Since $\mathrm{Bd}(A)$ is the intersection of two closed sets, it follows from (C3) that $\mathrm{Bd}(A)$ is closed. ∎

Problem 4.5

Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. Prove that

$$\mathrm{Bd}(A) = \mathrm{Bd}(A^c).$$

4.3 Examples

We now examine the interior, exterior and boundary of a set for some of the topologies that we have met.

Euclidean line

We begin by considering $\mathbb{R}$ with its Euclidean topology.

We have shown that $[0, 1]$ has interior $(0, 1)$ and exterior $(-\infty, 0) \cup (1, \infty)$. Worked problem 4.2. Similar techniques can be used to show that the intervals $[a, b]$, $[a, b)$, $(a, b]$ and (a, b) all have interior (a, b) and exterior $(-\infty, a) \cup (b, \infty)$. The only points that are not in the interior or the exterior are the points a and b, and so, by Theorem 4.5, the boundary of each of these intervals is $\{a, b\}$.

Similarly, we can show that the infinite intervals $(-\infty, a)$ and $(-\infty, a]$ have interior $(-\infty, a)$ and exterior (a, ∞), and the intervals (a, ∞) and $[a, \infty)$ have interior (a, ∞) and exterior $(-\infty, a)$. Thus, by Theorem 4.5, the boundary of each of these intervals is $\{a\}$.

We know that any one-point set $\{a\}$ is closed and not open. The interior of $\{a\}$ is contained in $\{a\}$. It cannot equal $\{a\}$, since Corollary 4.3 would then imply that $\{a\}$ is open. Thus the interior of $\{a\}$ is $\varnothing$. Corollary 4.3 also implies that the exterior of $\{a\}$ is equal to $\{a\}^c = (-\infty, a) \cup (a, \infty)$. It follows from Theorem 4.5 that the boundary of $\{a\}$ is the set $\{a\}$ itself.

Indiscrete and discrete topologies

Problem 4.6

Let X be a set and let A be a non-empty proper subset. Find the interior, exterior and boundary of A when:

(a) X carries the indiscrete topology;

(b) X carries the discrete topology.

Deleted-point topology

Let $\mathcal{T}_a$ be the a-deleted-point topology on a set X with at least two points, and let A be a non-empty proper subset of X. There are two cases to consider, depending on whether $a \in A$ or $a \notin A$.

Recall that a set is in $\mathcal{T}_a$ if it is X or does not contain a.

Consider the case when $a \in A$. It follows from Worked problem 1.1 that A is closed. So, by Corollary 4.3,

$$\text{Ext}(A) = A^c.$$

Any proper subset $B \subset X$ is open if and only if $a \notin B$, so $A - \{a\}$ is open, and any other open subset of A must be a subset of this. Thus, by Theorem 4.2,

$$\text{Int}(A) = A - \{a\}.$$

Finally, by Theorem 4.5,

$$\text{Bd}(A) = (\text{Ext}(A) \cup \text{Int}(A))^c = (A^c \cup (A - \{a\}))^c = (X - \{a\})^c = \{a\}.$$

We ask you to consider the case when $a \notin A$ in the following problem.

Problem 4.7

Let X be a set containing at least two points, let $a \in X$, and let $\mathcal{T}_a$ denote the a-deleted-point topology on X. Show that, if $A \subset X$ with $a \notin A$, then

$$\text{Int}(A) = A, \quad \text{Ext}(A) = A^c - \{a\}, \quad \text{Bd}(A) = \{a\}.$$

Co-finite topology

Consider the co-finite topology on an infinite set X and let $A \subseteq X$. We have seen in Subsection 2.4 that, if A is finite then $\text{Cl}(A) = A$, and if A is infinite then $\text{Cl}(A) = X$.

Recall that a non-empty set is open for the co-finite topology if and only if its complement is finite.

We now find the interior, exterior and boundary of A.

Finite subsets

If A is finite, then, by definition of the co-finite topology, the only open subset that it contains is $\varnothing$. So, by Theorem 4.2,

$$\text{Int}(A) = \varnothing.$$

Since $\text{Cl}(A) = A$, it follows from Corollary 4.4 that

$$\text{Ext}(A) = A^c,$$

which is infinite.

Finally, by Theorem 4.5,

$$\text{Bd}(A) = (\text{Ext}(A) \cup \text{Int}(A))^c = (A^c \cup \varnothing)^c = (A^c)^c = A.$$

Infinite subsets

If A is infinite, then there are two cases to consider, depending on whether A is open or not.

Suppose A is *open*.

By Corollary 4.3,

$$\text{Int}(A) = A.$$

Since A is infinite, $\text{Cl}(A) = X$ and so, by Corollary 4.4,

$$\text{Ext}(A) = X^c = \varnothing.$$

So A^c is finite.

Finally, by Theorem 4.5,

$$\mathrm{Bd}(A) = (\mathrm{Ext}(A) \cup \mathrm{Int}(A))^c = (\varnothing \cup A)^c = A^c,$$

which is a finite set.

Suppose A is *not open*.

By definition of the co-finite topology, the complement of A is infinite. So, if B is any non-empty subset of A, then B^c is also infinite, and so B cannot be open. We conclude that the only open set contained in A is the empty set. Thus, by Theorem 4.2,

$$\mathrm{Int}(A) = \varnothing.$$

However, since A is infinite, $\mathrm{Cl}(A) = X$ and so, by Corollary 4.4,

$$\mathrm{Ext}(A) = X^c = \varnothing.$$

Finally, by Theorem 4.5,

$$\mathrm{Bd}(A) = (\mathrm{Ext}(A) \cup \mathrm{Int}(A))^c = (\varnothing \cup \varnothing)^c = \varnothing^c = X.$$

4.4 The algebra of interiors, exteriors and boundaries

We turn now to some general results about the behaviour of interiors, exteriors and boundaries under unions and intersections.

First, we clarify the relationship between closure, interior, exterior and boundary.

Theorem 4.7

Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. Then

$$\mathrm{Cl}(A) = \mathrm{Int}(A) \cup \mathrm{Bd}(A) \quad \text{and} \quad \mathrm{Cl}(A^c) = \mathrm{Ext}(A) \cup \mathrm{Bd}(A).$$

Proof We first show that $\mathrm{Cl}(A) = \mathrm{Int}(A) \cup \mathrm{Bd}(A)$.

By Theorem 4.5,

$$\mathrm{Ext}(A) \cup \mathrm{Int}(A) \cup \mathrm{Bd}(A) = X.$$

Thus, since $\mathrm{Ext}(A)$, $\mathrm{Int}(A)$ and $\mathrm{Bd}(A)$ are mutually disjoint,

$$\mathrm{Int}(A) \cup \mathrm{Bd}(A) = \mathrm{Ext}(A)^c = \mathrm{Cl}(A),$$

by Corollary 4.4.

See Remark (ii) following Theorem 4.5.

We ask you to verify that $\mathrm{Cl}(A^c) = \mathrm{Ext}(A) \cup \mathrm{Bd}(A)$ in the following problem. ∎

Problem 4.8 _____

Let $(X, \mathcal{T})$ be a topological space and let $A \subseteq X$. Prove that

$$\mathrm{Cl}(A^c) = \mathrm{Ext}(A) \cup \mathrm{Bd}(A).$$

Theorem 4.7 tells us that the closure of a set A consists of the interior of A together with the boundary of A. So, by Theorem 2.4, a set is closed if and only if it contains all of its boundary points.

Let us now see what happens to interiors, exteriors and boundaries under unions and intersections.

Theorem 4.8

Let $(X, \mathcal{T})$ be a topological space and let $A, B \subseteq X$. Then:

(a) $\operatorname{Int}(A) \cup \operatorname{Int}(B) \subseteq \operatorname{Int}(A \cup B)$;

(b) $\operatorname{Ext}(A) \cup \operatorname{Ext}(B) \subseteq \operatorname{Ext}(A \cap B)$;

(c) $\operatorname{Int}(A) \cap \operatorname{Int}(B) = \operatorname{Int}(A \cap B)$;

(d) $\operatorname{Ext}(A) \cap \operatorname{Ext}(B) = \operatorname{Ext}(A \cup B)$.

Proof

Since both $A \subseteq A \cup B$ and $B \subseteq A \cup B$, it follows that

(a) $\operatorname{Int}(A) \subseteq \operatorname{Int}(A \cup B)$ and $\operatorname{Int}(B) \subseteq \operatorname{Int}(A \cup B)$. Hence,

$$\operatorname{Int}(A) \cup \operatorname{Int}(B) \subseteq \operatorname{Int}(A \cup B).$$

(b) Since A and B are arbitrary subsets of X, we also have

$$\operatorname{Int}(A^c) \cup \operatorname{Int}(B^c) \subseteq \operatorname{Int}(A^c \cup B^c).$$

Using De Morgan's Second Law, this gives

$$\operatorname{Int}(A^c) \cup \operatorname{Int}(B^c) \subseteq \operatorname{Int}((A \cap B)^c).$$

It now follows from Lemma 4.1 that

$$\operatorname{Ext}(A) \cup \operatorname{Ext}(B) \subseteq \operatorname{Ext}(A \cap B).$$

(c) Similarly, since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, we obtain

$$\operatorname{Int}(A \cap B) \subseteq \operatorname{Int}(A) \cap \operatorname{Int}(B).$$

$\operatorname{Int}(A) \cap \operatorname{Int}(B)$ is the intersection of two open sets and hence is open.
Moreover, $\operatorname{Int}(A) \subseteq A$ and $\operatorname{Int}(B) \subseteq B$, so $\operatorname{Int}(A) \cap \operatorname{Int}(B) \subseteq A \cap B$.
Since, by Theorem 4.2, $\operatorname{Int}(A \cap B)$ is the largest open subset of $A \cap B$,
it follows that

$$\operatorname{Int}(A) \cap \operatorname{Int}(B) \subseteq \operatorname{Int}(A \cap B).$$

Therefore

$$\operatorname{Int}(A) \cap \operatorname{Int}(B) = \operatorname{Int}(A \cap B).$$

(d) Since A and B are arbitrary subsets of X, we also have

$$\operatorname{Int}(A^c) \cap \operatorname{Int}(B^c) = \operatorname{Int}(A^c \cap B^c).$$

Using De Morgan's First Law, this gives

$$\operatorname{Int}(A^c) \cap \operatorname{Int}(B^c) = \operatorname{Int}((A \cup B)^c).$$

It now follows from Lemma 4.1 that

$$\operatorname{Ext}(A) \cap \operatorname{Ext}(B) = \operatorname{Ext}(A \cup B).$$ ∎

Problem 4.9

On the Euclidean line, let $A = [0, 1]$ and $B = (1, 2]$. Show that:

(a) $\operatorname{Int}(A) \cup \operatorname{Int}(B)$ is a *proper* subset of $\operatorname{Int}(A \cup B)$;

(b) $\operatorname{Ext}(A) \cup \operatorname{Ext}(B)$ is a *proper* subset of $\operatorname{Ext}(A \cap B)$.

Corollary 4.9

Let $(X, \mathcal{T})$ be a topological space and let $A, B \subseteq X$. Then

$$\mathrm{Bd}(A \cup B) \subseteq \mathrm{Bd}(A) \cup \mathrm{Bd}(B).$$

Proof We use the characterization of the boundary in terms of closure, given in Corollary 4.6, together with the behaviour of closures under unions and intersections given in Theorem 2.6.

We can also prove this result by using the results of Theorem 4.8, which is why we classify it as a corollary to that theorem.

$$
\begin{aligned}
\mathrm{Bd}(A \cup B) &= \mathrm{Cl}(A \cup B) \cap \mathrm{Cl}((A \cup B)^c), \quad \text{by Corollary 4.6,} \\
&= \mathrm{Cl}(A \cup B) \cap \mathrm{Cl}(A^c \cap B^c), \quad \text{by De Morgan's First Law,} \\
&\subseteq (\mathrm{Cl}(A) \cup \mathrm{Cl}(B)) \cap (\mathrm{Cl}(A^c) \cap \mathrm{Cl}(B^c)), \quad \text{by Theorem 2.6,} \\
&= (\mathrm{Cl}(A) \cap \mathrm{Cl}(A^c) \cap \mathrm{Cl}(B^c)) \cup (\mathrm{Cl}(B) \cap \mathrm{Cl}(A^c) \cap \mathrm{Cl}(B^c)), \\
&\qquad \text{by Theorem 2.6 of } \textit{Unit A3,} \\
&\subseteq (\mathrm{Cl}(A) \cap \mathrm{Cl}(A^c)) \cup (\mathrm{Cl}(B) \cap \mathrm{Cl}(B^c)) \\
&= \mathrm{Bd}(A) \cup \mathrm{Bd}(B), \quad \text{by Corollary 4.6.} \quad \blacksquare
\end{aligned}
$$

Problem 4.10 _____

Let $(X, \mathcal{T})$ be a topological space and let $A, B \subseteq X$. Use Corollary 4.9 and the result of Problem 4.5 to show that

$$\mathrm{Bd}(A \cap B) \subseteq \mathrm{Bd}(A) \cup \mathrm{Bd}(B).$$

4.5 Homeomorphisms

A *topological invariant* is a property of a topological space that is preserved under homeomorphisms.

We saw in *Unit A3* that openness is preserved under homeomorphisms, so openness is a topological invariant.

Problem 4.11 _____

Show that closedness is a topological invariant.

We conclude this unit by showing that closure, interior, exterior and boundary are all topological invariants. This should not be a surprise as they can all be described in terms of open and closed sets, which we have just seen are topological invariants.

Theorem 4.10

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, let $f \colon X \to Y$ be a homeomorphism and let $A \subseteq X$. Then:

(a) $f(\mathrm{Cl}(A)) = \mathrm{Cl}(f(A))$;

(b) $f(\mathrm{Ext}(A)) = \mathrm{Ext}(f(A))$;

(c) $f(\mathrm{Int}(A)) = \mathrm{Int}(f(A))$;

(d) $f(\mathrm{Bd}(A)) = \mathrm{Bd}(f(A))$.

Proof We prove only (a); the proofs of the other statements are similar.

We first show that $f(\mathrm{Cl}(A)) \subseteq \mathrm{Cl}(f(A))$.

By Lemma 2.3, $\mathrm{Cl}(f(A))$ is a closed set. Since f is a homeomorphism, it is continuous. It follows from Theorem 1.5 that $f^{-1}(\mathrm{Cl}(f(A)))$ is a closed set. Moreover, by Theorem 2.2,

$$\mathrm{Cl}(f(A)) \supseteq f(A),$$

so that

$$f^{-1}(\mathrm{Cl}(f(A))) \supseteq f^{-1}(f(A)) = A.$$

Hence $f^{-1}(\mathrm{Cl}(f(A)))$ is a closed set that contains A, and so, by Theorem 2.5,

$$\mathrm{Cl}(A) \subseteq f^{-1}(\mathrm{Cl}(f(A))).$$

Thus

$$f(\mathrm{Cl}(A)) \subseteq \mathrm{Cl}(f(A)).$$

We now show that $\mathrm{Cl}(f(A)) \subseteq f(\mathrm{Cl}(A))$.

Since f is a homeomorphism, f^{-1} exists and is continuous. Thus, repeating the above argument with f replaced by f^{-1} and A by $f(A)$, we obtain

$$f^{-1}(\mathrm{Cl}(f(A))) \subseteq \mathrm{Cl}(f^{-1}(f(A))).$$

This gives

$$f^{-1}(\mathrm{Cl}(f(A))) \subseteq \mathrm{Cl}(A),$$

and so

$$\mathrm{Cl}(f(A)) \subseteq f(\mathrm{Cl}(A)).$$

Thus $\mathrm{Cl}(f(A)) = f(\mathrm{Cl}(A))$. ∎

Problem 4.12

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces and let $f : X \to Y$ be a homeomorphism. Prove that $f(\mathrm{Int}(A)) = \mathrm{Int}(f(A))$.

Conclusion

This unit completes Block A. You have come a considerable way since the beginning of this block, which began by extending ideas from real analysis that you met in other courses. You then went on to study metric spaces, topological spaces, and finally the internal structure of sets from a topological point of view. You should now have an increased understanding of what it means for a function to be continuous and should be able to define a notion of continuity between arbitrary sets, provided that topologies have been specified.

When we take up the study of general topology again in Block C, we shall consider new types of sets — for example, connected and compact sets. In particular, we shall discover that the closed bounded interval $[a, b]$ is connected (which is why the Intermediate Value Theorem holds) and that it is also compact (which is why the functions in $C[a, b]$ have maxima and minima and attain them both).

Block B concerns the structure of surfaces, under the name *geometric topology*. Many of the concepts that you have met in these first four units serve to place the theory of surfaces on a firm foundation. In particular, the language of homeomorphisms allows us to consider as equivalent those surfaces that are continuous, reversible, deformations of each other — topological properties of one such surface are reflected in any homeomorphic surface.

Solutions to problems

1.1 **(a)** **(i)** $A^c = \mathbb{R} - [0, \infty) = (-\infty, 0)$.

(ii) $A^c = \mathbb{R} - (0, 2] = (-\infty, 0] \cup (2, \infty)$.

(iii) $A^c = \mathbb{R} - ([0, 1] \cup [2, 3])$

$\qquad = (-\infty, 0) \cup (1, 2) \cup (3, \infty)$.
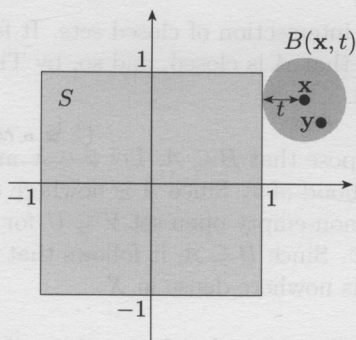
(b) **(i)** $(-\infty, 0)$ is an open set.

(ii) $(-\infty, 0] \cup (2, \infty)$ is not an open set.

(iii) $(-\infty, 0) \cup (1, 2) \cup (3, \infty)$ is an open set, since it is the union of open intervals.

1.2 Suppose that $\mathbf{x} = (x_1, x_2) \in S^c = \mathbb{R}^2 - S$. Then either $|x_1| > 1$ or $|x_2| > 1$ (or both). Let $t = \max\{|x_1| - 1, |x_2| - 1\}$, so that $t > 0$. Now suppose that $\mathbf{y} = (y_1, y_2) \in B(\mathbf{x}, t)$. Then at least one of $|y_1|$ or $|y_2|$ is strictly larger than 1. Hence

$$B(\mathbf{x}, t) \subseteq S^c$$

and so S^c is an open set.

1.3 **(a)** The closed sets are the complements of the sets in $\mathcal{T}$. These are the sets X, $\{b, c\}$, $\{a\}$ and $\varnothing$.

(b) The closed sets are the same as the open sets and so all the sets in $\mathcal{T}$ are both open and closed.

(c) The subsets of X that are neither open nor closed are those that do not belong to $\mathcal{T}$. These are the sets $\{b\}, \{c\}, \{a, b\}$ and $\{a, c\}$.

1.4 The complement of $\varnothing$ with respect to X is X. By (T1), this is an open set, so $\varnothing$ is closed.

The complement of X with respect to X is $\varnothing$. By (T1), this is an open set, so X is closed.

1.5 Let $A \subseteq X$. Then, by Theorem 1.1, A is closed if and only if A^c is open.

(a) Since, by definition, every subset is open for the discrete topology, A^c is open, and so A is closed. So, every subset of X is also closed for the discrete topology.

(b) For the indiscrete topology, A^c is open if and only if $A^c = \varnothing$ or $A^c = X$. Hence A is closed if and only if $A = \varnothing^c = X$ or $A = X^c = \varnothing$.

1.6 By Worked problem 1.1, the non-empty closed sets are those sets that contain the point a. So $\{a\}$ is closed. If $x \neq a$, then $a \notin \{x\}$ and so $\{x\}$ is not closed. Thus the only closed one-point subset of X is $\{a\}$.

1.7 Let $D \subseteq X$ be closed. The set D is closed if and only if D^c is open. For the co-finite topology, D^c is open if and only if $D^c = \varnothing$ or $(D^c)^c = D$ is finite. Hence, the collection of closed subsets of X is

$$\{X\} \cup \{D \subseteq X : D \text{ is finite}\}.$$

1.8 **(a)** Suppose that $D \subseteq A$ is $\mathcal{T}$-closed. Then $D \cap A = D$, and so D can be written as the intersection of a $\mathcal{T}$-closed set with A. Hence D is $\mathcal{T}_A$-closed.

(b) There are many possible examples. For instance, the set A itself is always $\mathcal{T}_A$-closed but may not be $\mathcal{T}$-closed. Let $X = \mathbb{R}$ and let $\mathcal{T}$ be the Euclidean topology on $\mathbb{R}$; then, for example, $A = (0, 1)$ is not $\mathcal{T}$-closed but is $\mathcal{T}_A$-closed.

1.9 Let $\mathcal{C}$ be a collection of subsets of X satisfying (C1)–(C3). We show that $\mathcal{C}^{(c)}$ satisfies (T1)–(T3).

(T1) We know from (C1) that $\varnothing$ and X belong to $\mathcal{C}$. Thus $\varnothing^c = X \in \mathcal{C}^{(c)}$ and $X^c = \varnothing \in \mathcal{C}^{(c)}$.

So (T1) is satisfied.

(T2) Let $U_1, U_2 \in \mathcal{C}^{(c)}$ and let $U = U_1 \cap U_2$. We show that $U \in \mathcal{C}^{(c)}$.

By De Morgan's Second Law,

$$U^c = (U_1 \cap U_2)^c = U_1^c \cup U_2^c.$$

Thus U^c is the union of two sets in $\mathcal{C}$. By (C2), $U^c \in \mathcal{C}$, and hence $U = (U^c)^c \in \mathcal{C}^{(c)}$.

Thus (T2) is satisfied.

(T3) Let $\mathcal{F} = \{U_i : i \in I\}$ be a collection of sets in $\mathcal{C}^{(c)}$ and let $U = \bigcup_{i \in I} U_i$. We show that $U \in \mathcal{C}^{(c)}$.

By De Morgan's First Law,

$$U^c = \left(\bigcup_{i \in I} U_i \right)^c = \bigcap_{i \in I} U_i^c.$$

Since $U_i^c \in \mathcal{C}$, for each $i \in I$, it follows from (C3) that $U^c \in \mathcal{C}$ and so $U = (U^c)^c \in \mathcal{C}^{(c)}$.

Thus (T3) is satisfied.

Since $\mathcal{C}^{(c)}$ satisfies (T1)–(T3), it is a topology on X.

2.1 A set U belongs to $\mathcal{T}_a$ if and only if $U = X$ or $a \notin U$.

(a) For the point a, the only neighbourhood is X.

(b) If $x \neq a$, then a neighbourhood of x is either X or a proper subset $U \subset X$ with $x \in U$ and $a \notin U$.

2.2 Since $\mathbb{Q} \subseteq \text{Cl}(\mathbb{Q})$, irrespective of the topology, we need only investigate whether the irrationals are closure points of $\mathbb{Q}$.

(a) For the indiscrete topology, the only open sets are $\varnothing$ and $\mathbb{R}$. So, if $x \in \mathbb{Q}^c$, then $\mathbb{R}$ is the only neighbourhood of x. Since $\mathbb{R} \cap \mathbb{Q} \neq \varnothing$, it follows that x is a closure point of $\mathbb{Q}$. So $\text{Cl}(\mathbb{Q}) = \mathbb{R}$, which is both open and closed.

(b) For the 0-deleted-point topology, the open sets are $\mathbb{R}$ and sets not containing 0. If $x \in \mathbb{Q}^c$, then $x \neq 0$ and so $\{x\}$ is a neighbourhood of x that does not meet $\mathbb{Q}$. Thus x is not a closure point of $\mathbb{Q}$. Hence, for this topology, no irrational number is a closure point of $\mathbb{Q}$, and so $\text{Cl}(\mathbb{Q}) = \mathbb{Q}$.

Now $0 \in \mathbb{Q}$ and so $\text{Cl}(\mathbb{Q}) = \mathbb{Q}$ is not open. However, $\mathbb{Q}^c$ is open (since $0 \notin \mathbb{Q}^c$) and so $\text{Cl}(\mathbb{Q}) = \mathbb{Q}$ is closed.

2.3 Since $A \cap B \subseteq A$, we have, by Theorem 2.2, $\text{Cl}(A \cap B) \subseteq \text{Cl}(A)$. Similarly $\text{Cl}(A \cap B) \subseteq \text{Cl}(B)$. Thus

$$\text{Cl}(A \cap B) \subseteq \text{Cl}(A) \cap \text{Cl}(B).$$

2.4 By the definition of complements, $\mathbb{Q} \cap \mathbb{Q}^c = \varnothing$ and, by the definition of closure, $\text{Cl}(\varnothing) = \varnothing$. Thus

$$\text{Cl}(\mathbb{Q} \cap \mathbb{Q}^c) = \varnothing.$$

However, by Worked problem 2.1, $\text{Cl}(\mathbb{Q}) = \text{Cl}(\mathbb{Q}^c) = \mathbb{R}$, and so

$$\text{Cl}(\mathbb{Q}) \cap \text{Cl}(\mathbb{Q}^c) = \mathbb{R}.$$

Therefore, for $A = \mathbb{Q}$ and $B = \mathbb{Q}^c$, $\text{Cl}(A \cap B)$ is a proper subset of $\text{Cl}(A) \cap \text{Cl}(B)$.

2.5 Since (a_n) is an increasing sequence that is bounded above, it has a limit, a say.

By Theorem 2.2, $A \subseteq \text{Cl}(A)$, so it remains to determine which points in A^c belong to $\text{Cl}(A)$.

If $x \in A^c$ and $x \neq a$, then $x < a_1$, or $x > a$, or (since (a_n) is strictly increasing) there is an $n \in \mathbb{N}$ for which

$$a_n < x < a_{n+1}.$$

In each case, we can find $r > 0$ such that $(x - r, x + r) \cap A = \varnothing$, and so x is not a closure point of A.

If $x = a$, then, by the fried-egg property, any neighbourhood of x contains an interval $(a - r, a + r)$ for some $r > 0$. However small r is, since (a_n) converges to a, the definition of convergence of a sequence tells us that there is an $N \in \mathbb{N}$ such that $a - r < a_n < a + r$ for all $n > N$. Hence $(a - r, a + r) \cap A \neq \varnothing$. Thus any neighbourhood of a intersects A and so a is a closure point of A.

Therefore

$$\text{Cl}(A) = A \cup \{a\}.$$

2.6 Every point $y \in X$ has $d_0(x, y) \leq 1$. Hence $B_{d_0}[x, 1] = X$.

Recall (from the remark following Theorem 1.2) that one-point sets in a metric space are closed, so the open ball $B_{d_0}(x, 1) = \{x\}$ is closed. Therefore, by Theorem 2.4, $\text{Cl}(B_{d_0}(x, 1)) = \{x\}$.

Alternatively, if y is in $B_{d_0}(x, 1)^c$, then the set $B_{d_0}(y, 1) = \{y\}$ is a neighbourhood of y that does not meet $B_{d_0}(x, 1)$, and so y is not a closure point of $B_{d_0}(x, 1)^c$. Thus no point in $B_{d_0}(x, 1)^c$ is a closure point, and so $\text{Cl}(B_{d_0}(x, 1)) = B_{d_0}(x, 1) = \{x\}$.

2.7 Since U is non-empty and open with respect to the co-finite topology, U^c is finite, and hence so is $U^c \cap A$. Now, $A = (U \cap A) \cup (U^c \cap A)$. Since A is infinite and $U^c \cap A$ is finite, we deduce that $U \cap A$ is infinite.

2.8 As in the solution to Worked problem 2.4, we observe that

$$A = \{f \in C[0, 1] : f(x) = 0 \text{ for } x \in I\}$$
$$= \bigcap_{x \in I} \{f \in C[0, 1] : f(x) = 0\}.$$

This is an intersection of closed sets. It follows from (C3) that A is closed, and so, by Theorem 2.4, $\text{Cl}(A) = A$.

3.1 Suppose that $B \subseteq A$. Let U be a non-empty open subset of X. Since A is nowhere dense in X, there is a non-empty open set $V \subseteq U$ for which $V \cap A = \varnothing$. Since $B \subseteq A$, it follows that $V \cap B = \varnothing$, and so B is nowhere dense in X.

3.2 **(a)** Let A be an infinite subset of X. For A to be dense, every point in A^c must be a closure point of A, so let $x \in A^c$. Any neighbourhood U of x has a finite complement, and so $U^c \cap A$ is finite. Now, $A = (U \cap A) \cup (U^c \cap A)$. Since A is infinite and $U^c \cap A$ is finite, we deduce that $U \cap A \neq \varnothing$. Hence x is a closure point of A. So $\text{Cl}(A) = X$ — that is, A is dense in X.

(b) Let A be a finite subset of X and let U be a non-empty open subset of X. Put $V = U \cap A^c$.

Then $V \subseteq U$ and $V \cap A = \varnothing$.

Also, A^c is open (since $(A^c)^c = A$ is finite). So, by (T2), V is open.

Further, $V = U \cap A^c$ is non-empty, since (by De Morgan's First Law) $V^c = U^c \cup A$ is finite and X is infinite.

So A is nowhere dense in X.

3.3 Since U and V are open, by (T2) so is $U \cap V$. So we need only show that $U \cap V$ is dense — that is, $\mathrm{Cl}(U \cap V) = X$.

Let $x \in X$ and let W be a neighbourhood of x. Since U is dense in X, $W \cap U \neq \varnothing$. Moreover, $W \cap U$ is open. Let $y \in W \cap U$. Then $W \cap U$ is a neighbourhood of y. Since V is dense in X, it follows that $(W \cap U) \cap V \neq \varnothing$. But $(W \cap U) \cap V = W \cap (U \cap V)$, and so $W \cap (U \cap V) \neq \varnothing$. Since W is an arbitrary neighbourhood of x, we conclude that $x \in \mathrm{Cl}(U \cap V)$ and so $\mathrm{Cl}(U \cap V) = X$ as required.

3.4 By definition of the subspace topology, there exists an open set $U \subseteq X$ such that $U \cap A = \{a\}$. The set U is the required neighbourhood of a.

3.5 Let $A \subseteq X$. If $A = \varnothing$, then there is nothing to prove. So suppose $a \in A$. We know that every subset of X is open. In particular, $\{a\}$ is open. So an isolating neighbourhood for a is the open set $\{a\}$; thus a is an isolated point.

3.6 Let $x \in \mathbb{Q}^c$ and let U be a neighbourhood of x. Then there is a rational number $r > 0$ such that $(x - r, x + r) \subseteq U$. But the point $x + \frac{1}{2}r$ is irrational and lies in $(x - r, x + r)$, and so $\mathbb{Q}^c \cap U \neq \{x\}$. Since this holds for all neighbourhoods of x, we conclude that x is not isolated, and so $\mathbb{Q}^c$ has no isolated points. (Note that if r is irrational then $x + \frac{1}{2}r$ may or may not be irrational.)

4.1 Let $A = (0, 1) \cup \{2\}$.

Interior

Since $\mathrm{Int}(A) \subseteq A$, we need only decide which points of A are interior points. If $x \in (0, 1)$, then $(0, 1)$ is a neighbourhood of x that lies entirely within A. Hence every point of $(0, 1)$ is an interior point of A. For $x = 2$, each neighbourhood of x contains points that are not in A. Hence 2 is not an interior point of A. Thus

$$\mathrm{Int}(A) = (0, 1).$$

Exterior

We use the fact, from Lemma 4.1, that $\mathrm{Ext}(A) = \mathrm{Int}(A^c)$. Now

$$A^c = (-\infty, 0] \cup [1, 2) \cup (2, \infty).$$

If $x \in (-\infty, 0) \cup (1, 2) \cup (2, \infty)$, then $(-\infty, 0) \cup (1, 2) \cup (2, \infty)$ is a neighbourhood of x that is contained in A^c and so x is an interior point of A^c. If $x = 0$ or $x = 1$, then each neighbourhood of x has non-empty intersection with $(0, 1) \subseteq (A^c)^c = A$, and so x is not an interior point of A^c. Thus

$$\mathrm{Ext}(A) = \mathrm{Int}(A^c) = (-\infty, 0) \cup (1, 2) \cup (2, \infty).$$

4.2 Since $\varnothing$ and X are both open and closed, it follows from Corollary 4.3 that $\mathrm{Int}(\varnothing) = \varnothing$, $\mathrm{Ext}(\varnothing) = \varnothing^c = X$, $\mathrm{Int}(X) = X$ and $\mathrm{Ext}(X) = X^c = \varnothing$.

4.3 From the solution to Worked problem 2.2, $\mathrm{Cl}(A) = A \cup \{0\}$. Hence, by Corollary 4.4,

$$\mathrm{Ext}(A) = \mathrm{Cl}(A)^c$$
$$= (\{\tfrac{1}{n} : n \in \mathbb{N}\} \cup \{0\})^c$$
$$= (-\infty, 0) \cup \bigcup_{n \in \mathbb{N}} \left(\tfrac{1}{n+1}, \tfrac{1}{n} \right) \cup (1, \infty).$$

4.4 **(a)** By Worked problem 2.1(a), $\mathrm{Cl}(\mathbb{Q}) = \mathrm{Cl}(\mathbb{Q}^c) = \mathbb{R}$. It follows from Corollary 4.4 that

$$\mathrm{Int}(\mathbb{Q}) = \mathrm{Cl}(\mathbb{Q}^c)^c = \mathbb{R}^c = \varnothing,$$
$$\mathrm{Ext}(\mathbb{Q}) = \mathrm{Cl}(\mathbb{Q})^c = \mathbb{R}^c = \varnothing.$$

By Theorem 4.5,

$$\mathrm{Bd}(\mathbb{Q}) = (\mathrm{Ext}(\mathbb{Q}) \cup \mathrm{Int}(\mathbb{Q}))^c = \varnothing^c = \mathbb{R}.$$

(b) For $\mathbb{Q}^c$, similar arguments can be used to obtain

$$\mathrm{Int}(\mathbb{Q}^c) = \varnothing, \quad \mathrm{Ext}(\mathbb{Q}^c) = \varnothing, \quad \mathrm{Bd}(\mathbb{Q}^c) = \mathbb{R}.$$

4.5 By Corollary 4.6,

$$\mathrm{Bd}(A^c) = \mathrm{Cl}(A^c) \cap \mathrm{Cl}((A^c)^c)$$
$$= \mathrm{Cl}(A^c) \cap \mathrm{Cl}(A) = \mathrm{Bd}(A).$$

4.6 **(a)** For the indiscrete topology, the only open sets are $\varnothing$ and X. Since A is a *proper* subset of X, the only open set inside A is $\varnothing$. Thus, by Theorem 4.2, $\mathrm{Int}(A) = \varnothing$.

Since A^c is also a non-empty proper subset of X, similar arguments show that $\mathrm{Ext}(A) = \varnothing$.

By Theorem 4.5,

$$\mathrm{Bd}(A) = (\mathrm{Ext}(A) \cup \mathrm{Int}(A))^c = \varnothing^c = X.$$

(b) For the discrete topology, every subset of X is both open and closed. So A is open and, by Corollary 4.3, $\mathrm{Int}(A) = A$.

Also, A is closed and, by Corollary 4.3, $\mathrm{Ext}(A) = A^c$.

By Theorem 4.5,

$$\mathrm{Bd}(A) = (\mathrm{Ext}(A) \cup \mathrm{Int}(A))^c$$
$$= (A^c \cup A)^c = X^c = \varnothing.$$

4.7 Since $a \notin A$, A is open. By Corollary 4.3,

$$\mathrm{Int}(A) = A.$$

Now, by Worked problem 1.1, $A \cup \{a\}$ is the smallest closed set containing A and so, by Theorem 2.5, $\mathrm{Cl}(A) = A \cup \{a\}$. By Corollary 4.4,

$$\mathrm{Ext}(A) = \mathrm{Cl}(A)^c = (A \cup \{a\})^c = A^c - \{a\}.$$

By Theorem 4.5,

$$\mathrm{Bd}(A) = (\mathrm{Ext}(A) \cup \mathrm{Int}(A))^c = ((A^c - \{a\}) \cup A)^c$$
$$= (X - \{a\})^c = \{a\}.$$

4.8 By the first result of Theorem 4.7,

$$\mathrm{Cl}(A^c) = \mathrm{Int}(A^c) \cup \mathrm{Bd}(A^c).$$

By Lemma 4.1, $\mathrm{Int}(A^c) = \mathrm{Ext}(A)$ and, by Problem 4.5, $\mathrm{Bd}(A^c) = \mathrm{Bd}(A)$. Thus

$$\mathrm{Cl}(A^c) = \mathrm{Ext}(A) \cup \mathrm{Bd}(A).$$

4.9 We have $A \cup B = [0,2]$ and so $\text{Int}(A \cup B) = (0,2)$. Thus

$$\text{Int}(A) \cup \text{Int}(B) = (0,1) \cup (1,2)$$
$$= \text{Int}(A \cup B) - \{1\}.$$

Also, $A \cap B = \varnothing$ and so $\text{Ext}(A \cap B) = \mathbb{R}$. Thus

$$\text{Ext}(A) \cup \text{Ext}(B) = (-\infty, 0) \cup (1, \infty) \cup (-\infty, 1) \cup (2, \infty)$$
$$= (-\infty, 1) \cup (1, \infty)$$
$$= \text{Ext}(A \cap B) - \{1\}.$$

4.10 It follows from Problem 4.5 that

$$\text{Bd}(A \cap B) = \text{Bd}((A \cap B)^c).$$

Using De Morgan's Second Law, this gives

$$\text{Bd}(A \cap B) = \text{Bd}(A^c \cup B^c).$$

It now follows from Corollary 4.9 and Problem 4.5 that

$$\text{Bd}(A \cap B) \subseteq \text{Bd}(A^c) \cup \text{Bd}(B^c)$$
$$= \text{Bd}(A) \cup \text{Bd}(B).$$

4.11 Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces, let $D \subseteq X$ and let $f : X \to Y$ be a homeomorphism. We need to show that

$$D \text{ is closed if and only if } f(D) \text{ is closed.}$$

If D is closed, then since f is a homeomorphism, the inverse function $f^{-1} : Y \to X$ exists and is continuous. Hence, by Theorem 1.5, applied to f^{-1},

$$(f^{-1})^{-1}(D) = f(D) \text{ is closed.}$$

Conversely, if $f(D)$ is closed, then, since f is one-one,

$$D = f^{-1}(f(D)).$$

But f is continuous, and so by Theorem 1.5, D is closed.

4.12 First we show that $\text{Int}(f(A)) \subseteq f(\text{Int}(A))$. By Theorem 4.2, $\text{Int}(f(A))$ is an open set. Since f is a homeomorphism, it is continuous, and so $f^{-1}(\text{Int}(f(A)))$ is an open set. Moreover,

$$\text{Int}(f(A)) \subseteq f(A),$$

so that

$$f^{-1}(\text{Int}(f(A))) \subseteq f^{-1}(f(A)) = A.$$

Hence $f^{-1}(\text{Int}(f(A)))$ is an open set contained in A, and so, by Theorem 4.2,

$$f^{-1}(\text{Int}(f(A))) \subseteq \text{Int}(A).$$

Thus

$$\text{Int}(f(A)) \subseteq f(\text{Int}(A)).$$

We now show that $f(\text{Int}(A)) \subseteq \text{Int}(f(A))$. Since f is a homeomorphism, f^{-1} exists and is continuous. Thus, repeating the above argument with f replaced by f^{-1} and A by $f(A)$, we obtain

$$\text{Int}(f^{-1}(f(A))) \subseteq f^{-1}(\text{Int}(f(A))).$$

This gives

$$\text{Int}(A) \subseteq f^{-1}(\text{Int}(f(A))),$$

and so

$$f(\text{Int}(A)) \subseteq \text{Int}(f(A)).$$

Thus $f(\text{Int}(A)) = \text{Int}(f(A))$.

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